CONTINUITY OF THE SPECTRUM AND SPECTRAL RADIUS

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Abstract. Let $A$ be a Banach algebra containing an element $x$. Topological
conditions on the spectrum of $x$ are given which are necessary and sufficient to
ensure the continuity of the spectrum or spectral radius at $x$.

1. Introduction. In this paper $C$ denotes the field of complex numbers, and $K$ the
metric space of nonempty compact subsets of $C$ endowed with the Hausdorff
metric $\Delta$. If $K_1, K_2 \subseteq K$, then

$$\Delta(K_1, K_2) = \max \left( \sup_{\lambda \in K_2} d(\lambda, K_1), \sup_{\lambda \in K_1} d(\lambda, K_2) \right)$$

where $d(\lambda, K_i) = \inf_{\mu \in K_i} |\lambda - \mu|$.

$A$ will denote a Banach algebra, which we will always assume is unital without
loss of generality. For $x \in A$, $\sigma(x)$ and $r(x)$ denote the spectrum of $x$ and spectral
radius of $x$ respectively. We are interested in determining the points in $A$ at which
the spectrum $\sigma: A \to K$, $x \to \sigma(x)$ or the spectral radius $r: A \to [0, \infty)$, $x \to r(x)$ are continuous. Newburgh [5] initiated the study of this problem in 1951. A famous
elementary example, due to Kakutani [4, pp. 248–249], shows that the spectral radius is
discontinuous at certain elements in the $C^*$-algebra $B(H)$ of all bounded linear
operators on a separable Hilbert space $H$. Recently Conway and Morrel [3] have
given necessary and sufficient conditions for the continuity of $\sigma$ and $r$ at $T$ in
$B(H)$. Their results depend on a deep theorem of Apostol and Morrel [1, Theorem
3.1] a special case of which we now state, as it will be used in the sequel.

Theorem 1. If $T$ is a normal operator on $H$, and $S$ is a closed subset of $C$ which
meets all the components of $\sigma(T)$, then there is a sequence $T_n$ in $B(H)$ converging to
$T$ in norm for which $\sigma(T_n) \subseteq S$ ($n > 0$).

Let $A$ be a Banach algebra, $x \in A$, and $U$ an open subset of $C$. There are a
number of results, due essentially to Newburgh [5] which we shall need later.

Theorem 2. If $U \supseteq \sigma(x)$, then there exists $\delta > 0$ such that $\|y - x\| < \delta$ implies
that $U \supseteq \sigma(y)$.

(This property of $\sigma$ is referred to as upper semicontinuity.)
Theorem 3. If $U$ contains a component of $\sigma(x)$, then there exists $\delta > 0$ such that $\|y - x\| < \delta$ implies that $U$ contains a component of $\sigma(y)$.

It is trivial to see that $r$ is always upper semicontinuous, and that, since $r(x) = \Delta(\sigma(x), \{0\})$, if $\sigma$ is continuous at $x$, so is $r$. An interesting result due to Aupetit [2] states that $r$ is uniformly continuous on $A$ if and only if $A/\text{rad}(A)$ is commutative ($\text{rad}(A)$ denotes the Jacobson radical of $A$).

2. Continuity of $r$ and $\sigma$. Throughout this section $A$ is a (unital) Banach algebra, $x \in A$, and $T$ a normal operator on a separable infinite dimensional Hilbert space $H$. Let $K \in K$, and put $\alpha(K) = \sup \{\inf_{\omega \in \omega}|\lambda|: \omega$ is a component of $K\}$, and $r(K) = \sup_{\lambda \in K}|\lambda|$. So $r(K) > \alpha(K)$. We assume $\sigma(T) = K$.

Definition 1. The set $K$ is an $r$-set (resp. $\sigma$-set) if for every Banach algebra $A$, and $x$ in $A$ with $\sigma(x) = K$, the spectral radius $r$ (resp. the spectrum $\sigma$) is continuous at $x$ in $\mathbb{C}$. A well-known result of Newburgh [5] can be restated by saying that if $K$ is totally disconnected it is a $\sigma$-set.

(Note that this is true for the spectrum $\sigma(S)$ of a compact or Riesz operator $S$ on a Banach space $X$, so that $\sigma$ is continuous at $S$ in the Banach algebra $B(X)$ of bounded linear operators on $X$.)

Assume $K$ is a member of $K$.

Proposition 1. The following are equivalent statements:

(i) $K$ is an $r$-set;
(ii) $\alpha(K) = r(K)$;
(iii) $T$ is a point of continuity of $r$ in $B(H)$.

Proof. (ii) $\Rightarrow$ (i). If $\alpha(K) = r(K)$ and $K = \sigma(x)$ for $x$ in some Banach algebra $A$ then, for any $\varepsilon > 0$, let $U = \{\lambda \in C: |\lambda| > r(x) - \varepsilon\}$. As there is a component $\omega$ of $K$ with $\inf_{\lambda \in \omega}|\lambda| > \alpha(K) - \varepsilon = r(x) - \varepsilon$, so $U \supseteq \omega$, and hence by Theorem 3, there exists $\delta > 0$ such that $\|y - x\| < \delta$ implies that $U$ contains a component of $\sigma(y)$. Therefore $r(y) > r(x) - \varepsilon$. This proves the lower semicontinuity of $r$ at $x$. As upper semicontinuity is automatic, it follows that $r$ is continuous at $x$. Thus we have shown that $K$ is an $r$-set.

(i) $\Rightarrow$ (iii) by definition.

(iii) $\Rightarrow$ (ii). Assume that (iii) holds and that $\alpha(K) < r(K)$. (We can always construct a normal $T$ with $\sigma(T) = K$, simply by choosing a diagonal operator whose diagonal entries are dense in $K$.) Now every component $\omega$ of $K$ meets the set $\Delta = \{\lambda \in C: |\lambda| < \rho\}$ where $\rho$ is chosen such that $\alpha(K) < \rho < r(K)$. Hence by Theorem 1, there is a sequence $T_n$ in $B(H)$ converging to $T$ in norm with $\sigma(T_n) \subseteq \Delta$. Thus $r(T_n) < \rho$ ($n > 0$). But this is impossible by the continuity of $r$ at $T$. This contradiction shows $\alpha(K) = r(K)$, and so (iii) $\Rightarrow$ (ii). The proof of implication (iii) $\Rightarrow$ (ii) is a simplification of a special case of the proof of Theorem 2.6 in [3].

For $K \in K$, let $K_0 = \{\lambda \in K: \text{the component of } \lambda \text{ in } K = \{\lambda\}\}$. (Thus $K = K_0$ if and only if $K_0$ is totally disconnected.) As before, $T$ is a normal operator on $H$ and $\sigma(T) = K$. $K_0$ denotes the closure of $K_0$ in $C$. 

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Proposition 2. The following are equivalent statements:

(i) \( K \) is a \( \sigma \)-set;

(ii) \( K = \overline{K_0} \);

(iii) for each \( \varepsilon > 0 \) and for each \( \lambda \in K \), \( B(\lambda, \varepsilon) = \{ \mu \in \mathbb{C} : |\mu - \lambda| < \varepsilon \} \) contains a component of \( K \);

(iv) \( T \) is a point of continuity of \( \sigma \) in \( B(H) \).

Proof. That (iii) \( \Rightarrow \) (ii) follows from some elementary topology. (This fact is also pointed out in [3, p. 19].)

(ii) \( \Rightarrow \) (i). Let \( A \) be any Banach algebra, and suppose that \( \sigma(x) = K \). Then for each \( \varepsilon > 0 \), there exist \( \lambda_1, \ldots, \lambda_n \in K \) such that \( B(\lambda_i, \varepsilon/2) \cup \cdots \cup B(\lambda_n, \varepsilon/2) \supseteq K \). Hence as \( K_0 = K \), there exist \( \mu_i \in K_0 \) (\( i = 1, \ldots, n \)) such that \( |\lambda_i - \mu_i| < \varepsilon/2 \). Thus there exist \( \delta_i > 0 \) (\( i = 1, \ldots, n \)) such that \( \|y - x\| < \delta_i \) implies that \( B(\lambda_i, \varepsilon/2) \) contains a component of \( \sigma(y) \). This follows from Theorem 3, and the fact that \( \{ \mu_i \} \) is a component of \( \sigma(x) = K \). Thus if \( \delta = \min_{1 \leq i \leq n} \delta_i \) then for \( \|y - x\| < \delta \) and \( \lambda \in \sigma(x) \) we have \( |\lambda - \lambda_i| < \varepsilon/2 \) for some \( i \), \( 1 < i \leq n \), and, for each \( i \), \( |\lambda_i - \lambda_i'| < \varepsilon/2 \) for some \( \lambda_i' \) in \( \sigma(y) \), as \( B(\lambda_i, \varepsilon/2) \) contains a component of \( \sigma(y) \).

Hence \( |\lambda - \lambda_i'| < \varepsilon \), or \( d(\lambda, \sigma(y)) < \varepsilon \). Thus \( \sup_{\lambda \in \sigma(x)} d(\lambda, \sigma(y)) < \varepsilon \) for \( \|y - x\| < \delta \).

But by the upper semicontinuity property, there exists \( \delta_0 < \delta, \delta_0 > 0 \), such that \( \|y - x\| < \delta_0 \) implies that \( \sup_{\lambda \in \sigma(y)} d(\lambda, \sigma(x)) < \varepsilon \). Thus \( \Delta(\sigma(y), \sigma(x)) < \varepsilon \) for \( \|y - x\| < \delta_0 \), showing that \( \sigma \) is continuous at \( x \). Hence \( K \) is a \( \sigma \)-set. (This argument is a generalization of one due to Newburgh [5] proving that totally disconnected sets are \( \sigma \)-sets.)

(i) \( \Rightarrow \) (iv) by definition.

(iv) \( \Rightarrow \) (iii). Assume that \( T \) is a point of continuity of \( \sigma \) and let \( \varepsilon > 0 \) and \( \lambda \in K = \sigma(T) \). If \( B(\lambda, \varepsilon) \) does not contain a component of \( K \), then \( S = \mathbb{C} \setminus B(\lambda, \varepsilon) \) is a closed set meeting all the components of \( \sigma(T) \), so by Theorem 1, there is a sequence \( T_n \) in \( B(H) \) with \( T_n \) converging to \( T \) in norm, but \( \sigma(T_n) \subseteq S \) (\( n > 0 \)). Hence \( \Delta(\sigma(T_n), \sigma(T)) \to d(\lambda, \sigma(T)) \geq \varepsilon \) (\( n \to 0 \)). But as \( \sigma \) is continuous at \( T \), \( \Delta(\sigma(T_n), \sigma(T)) \to 0 \) (\( n \to \infty \)). This contradiction shows \( B(\lambda, \varepsilon) \) contains a component of \( \sigma(T) \), and so (iv) implies (iii). (This result is a simplified proof of a special case of Theorem 3.1 in [3].)

Finally, it is easy to exhibit \( K \in \mathbb{K} \) such that \( K_0 \neq K = \overline{K_0} \) and to exhibit \( K \) such that \( \sigma(K) = r(K) \) but no component of \( K \) lies on the circle \( \{ \lambda : |\lambda| = r(K) \} \). (For example take \( K = \{ 1 - 1/n : n > 1 \} \cup \{ x + iy : x + y = 1; x, y > 0 \} \).)

References

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