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ADMISSIBLE KERNELS FOR STARSHAPED SETS

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ABSTRACT. Steven Lay has posed the following interesting question: If D is a convex subset of $\mathbf{R}^d$, then is there a starshaped set $S \neq D$ in $\mathbf{R}^d$ whose kernel is D? Thus the problem is that of characterizing those convex sets which are admissible as the kernel of some nonconvex starshaped set in $\mathbf{R}^d$. Here Lay's problem is investigated for closed sets, and the following results are obtained: If D is a nonempty closed convex subset of $\mathbf{R}^2$, then D is the kernel of some planar set $S \neq D$ if and only if D contains no line. If D is a compact convex set in $\mathbf{R}^d$, then there is a compact set $S \neq D$ in $\mathbf{R}^d$ whose kernel is D.

1. Introduction. Let S be a subset of $\mathbf{R}^d$. Set S is said to be *starshaped* if and only if there is some point p in S such that $[p, x] \subseteq S$ for every x in S. The set of all such points p is called the (convex) kernel of S, denoted ker S. While many interesting results have been obtained for starshaped sets and their kernels [2], a new problem has been posed recently by Steven Lay [1]. If D is a convex subset of $\mathbf{R}^d$, then is there a starshaped set $S \neq D$ in $\mathbf{R}^d$ whose kernel is D? That is, can we characterize those convex sets which are admissible as the kernel of some nonconvex starshaped set in $\mathbf{R}^d$? The question may be answered quickly in various special cases. In particular, when D is not a $(d-1)$-flat and D has dimension less than d, it is not hard to construct an appropriate set S. Similarly, when D is a polytope, such an S exists. However, the problem for arbitrary D is both intriguing and challenging. Here we begin by considering a closed convex set D in the plane, and constructions for S are given when D contains no line. If D is closed and contains a line, then it is proved that no such set S exists. Finally, extensions from $\mathbf{R}^2$ to $\mathbf{R}^d$ are obtained with certain modifications.

The following definitions and terminology will be used throughout the paper. Let S be a subset of $\mathbf{R}^d$. A point x in S is said to be a *point of local convexity* of S if and only if there is some neighborhood N of x such that $N \cap S$ is convex. In case S fails to be locally convex at point q in S, then q is called a *point of local nonconvexity* (lnc point) of S. For S a 2-dimensional convex set, line L is said to *support* S if and only if L contains no interior point of S and the distance from L to S is zero. (Notice that such a line need not meet S.) The terms conv S, aff S, cl S, bdry S, and int S will denote the convex hull, affine hull, closure, boundary, and interior of set S, respectively. When S is convex, dim S will be the dimension of set

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S. For distinct points x and y, $\operatorname{dist}(x, y)$ will represent the distance from x to y, $L(x, y)$ will represent the line determined by x and y, and $R(x, y)$ will be the ray emanating from x through y.

2. The results in $\mathbf{R}^2$. We begin with the following theorem for compact convex sets in $\mathbf{R}^2$.

THEOREM 1. *Let D be any nonempty compact convex set in $\mathbf{R}^2$. Then there is a compact set $S \neq D$ in $\mathbf{R}^2$ with* $\ker S = D$.

PROOF. We begin by disposing of the cases for $\dim D < 2$. If D is a singleton set, let S be the union of two closed disks whose intersection is exactly D. If D is a segment $[x, y]$, choose x_0, y_0 on $L(x, y)$ with $x_0 < x < y < y_0$, choose $z \notin L(x, y)$, and let $S = \operatorname{conv}\{x, y, z\} \cup [x_0, y_0]$.

Throughout the remainder of the argument, we assume that D is a full 2-dimensional set. For convenience of notation, we suppose that D lies in the unit disk $\{z: |z| = 1\}$, with the origin θ interior to D. Moreover, assume that D is oriented so that it has a horizontal line of support which intersects the set on the positive y axis. Let T_L and T_R denote the vertical lines which support D; T_L in the left open halfplane, T_R in the right open halfplane. For each point $x \notin D$, there are two distinct segments (x, y_1) and (x, y_2) disjoint from D whose lines support D at y_1 and y_2, respectively. We call $[x, y_1]$ the left supporting segment from x to D and call $[x, y_2]$ the right supporting segment from x to D provided that the order along bdry conv (y_1, x, y_2) in a clockwise direction from y_1 gives y_1, x, y_2. A parallel definition holds for left and right supporting lines $L(x, y_1)$ and $L(x, y_2)$.

We will construct set S inductively, using an increasing sequence of closed sets. Choose a point p_1 not in D and in the open convex region bounded by T_L and T_R. For the sake of simplicity, let p_1 be a point on the y axis for $y > 2$. Define $S_1 = \operatorname{conv}(p_1 \cup D)$. Let w_1 be the midpoint of the left supporting segment $[p_1, s_1]$ from p_1 to D. Let T_1 be the right supporting line from w_1 to D at point s_2. Choose the point u_1 of T_1 such that w_1 is between u_1 and s_2 and $\operatorname{dist}(w_1, u_1)$ is half the distance from w_1 to T_L. Then $(w_1, u_1) \cap S_1 = \varnothing$. Relabel peak p_1 by p_{21}, peak u_1 by $p_{22} \equiv b_2$, and define $S_2 = \bigcup \{\operatorname{conv}(p_{2i} \cup D): 1 \leq i \leq 2\}$. Label the lnc point w_1 of S_2 by s_{21}, and let s_{22} be the point of D such that $[p_{22}, s_{22}]$ is the left supporting segment from p_{22} to D.

Inductively, for $k > 2$ assume that consecutive peaks $p_{k1}, p_{k2}, \ldots, p_{kj} \equiv b_k$ are defined, $j = 2^{k-1}$, that $S_k = \bigcup \{\operatorname{conv}(p_{ki} \cup D): 1 \leq i \leq j\}$, and that $s_{k1}, s_{k2}, \ldots, s_{kj}$ are defined. We obtain S_{k+1} as follows. For each i, $1 \leq i \leq j$, let w_{ki} be the midpoint of segment $[p_{ki}, s_{ki}]$. Let T_{ki} be the right supporting line from w_{ki} to D. For $i \neq j$, choose point u_{ki} of T_{ki} whose distance to w_{ki} is half the distance from w_{ki} to segment $[p_{ki}, p_{k,i+1}]$ along T_{ki} and $(w_{ki}, u_{ki}) \cap S_k = \varnothing$. For $i = j$, choose point u_{kj} of T_{kj} such that $\operatorname{dist}(w_{kj}, u_{kj}) = \frac{1}{2}\operatorname{dist}(w_{kj}, T_L)$. (Recall that T_L is the left vertical line of support to D.) Relabel peaks $p_{k1}, u_{k1}, p_{k2}, u_{k2}, \ldots, p_{kj}, u_{kj}$ by $p_{k+1,1}, p_{k+1,2}, \ldots, p_{k+1,2j} \equiv b_{k+1}$, respectively, and refer to $p_{k+1,i}$ and $p_{k+1,i+1}$ as consecutive peaks, $1 \leq i \leq 2j - 1$. Define $S_{k+1} = \bigcup \{\operatorname{conv}(p_{k+1,i} \cup D): 1 \leq i \leq 2j\}$. For future reference, notice that for each peak $p_{k+1,i}$ of S_{k+1} and for M a

supporting line from $p_{k+1,i}$ to D, sets $\{p_{k+1,1}, \ldots, p_{k+1,i-1}\}$ and $\{p_{k+1,i+1}, \ldots, p_{k+1,2j}\}$ lie in opposite open halfplanes determined by M. Thus no point $p_{k+1,i}$ is captured by a remaining convex hull $\operatorname{conv}(p_{k+1,m} \cup D)$, $m \neq i$. Note also that the slopes of the right supporting lines to D from consecutive peaks of S_{k+1} form an increasing sequence, as do the slopes of the left supporting lines. Finally, observe that $(p_{k+1,i}, p_{k+1,i+1}) \cap S_{k+1} = \varnothing$ for $1 \leqslant i \leqslant 2j - 1$.

Continuing our inductive construction, label the lnc points of S_{k+1} by $s_{k+1,1}, s_{k+1,2}, \ldots, s_{k+1,2j-1}$, where $s_{k+1,i}$ is the lnc point of $\operatorname{conv}(p_{k+1,i} \cup D) \cup \operatorname{conv}(p_{k+1,i+1} \cup D)$, $1 \leqslant i \leqslant 2j - 1$. And let $s_{k+1,2j}$ be the point of D such that $[p_{k+1,2j}, s_{k+1,2j}]$ is the left supporting segment from $p_{k+1,2j} \equiv b_{k+1}$ to D. By our inductive construction, S_n is defined for every natural number n. We define set $S_L = \bigcup_{1 \leqslant n} S_n$.

Now begin again with set $S_1 \equiv S_1'$ and apply the construction above to the right halfplane, using right supporting segments (instead of left), left supporting lines (instead of right), and line T_R (instead of T_L). We thereby obtain an increasing sequence of sets $\{S_n'\}$, and define set $S_R = \bigcup_{1 \leqslant n} S_n'$. Finally, define $S = \operatorname{cl}(S_L \cup S_R)$. Observe that S_R contributes no points to S_L in the left halfplane, S_L contributes none to S_R in the right.

Clearly S is a compact set and $S \neq D$. We assert that $\ker S = D$. It is easy to show that $D \subseteq \ker S$, so we will establish only the reverse inclusion. As a preliminary step, we show that for p one of the peaks defined above, say $p = p_{ki}$, p is an lnc point of S. Without loss of generality, assume that $p \in S_L$. Let N be any circular neighborhood of p. By our construction, N contains another peak, say $q = p_{lm}$, where $l > k$ and where q and θ are in opposite open halfplanes determined by line $L(p_{ki}, s_{ki})$. Moreover, q may be selected so that p and q are consecutive peaks for S_l. Then by a previous comment, $(p, q) \cap S_l = \varnothing$. Select r so that p, r, q are consecutive peaks for S_{l+1}. For $n \geqslant l + 1$, every peak $t \neq p, r, q$ of S_n which lies in the convex region determined by rays $R(\theta, p)$ and $R(\theta, q)$ must lie in the open halfplane determined by $L(p, r)$ and containing θ and in the open halfplane determined by $L(r, q)$ and containing θ. Thus $\operatorname{conv}(t \cup D)$ is disjoint from $\operatorname{conv}(p, r, q)$, and such a peak t cannot contribute any point to $\operatorname{conv}(p, r, q)$ in S_n or in S. Using previous comments, it is easy to see that the remaining peaks of S_n cannot contribute any point to $\operatorname{conv}(p, r, q)$ either, and we conclude that $(p, q) \cap S = \varnothing$. Hence p is indeed an lnc point for S, and our preliminary result is established.

Now we are able to show that $\ker S \subseteq D$. Let $x \in S \sim D$ to prove that $x \notin \ker S$. Without loss of generality, assume that x is in the closed left halfplane. Recall that, for each $k \geqslant 2$, b_k denotes the last peak of S_k in our ordering, and clearly the sequence $\{b_k\}$ converges to a point of $D \cap T_L$. By our construction, it is not hard to see that $T_L \cap S = T_L \cap D$, so $x \notin T_L$. Therefore, there must be some peak p in the open left halfplane such that the right supporting segment from p to D has slope greater than the slope of the right supporting segment E from x to D. That is, x and θ lie in opposite open halfplanes E_1 and E_2, respectively, determined by the line of E. Again by our construction, there is a sequence of

distinct peaks in E_1 converging to a point of E, and it is not hard to see that this sequence is eventually in int conv$(p \cup x \cup D)$. Hence for n sufficiently large, int conv$(p \cup x \cup D)$ necessarily contains peaks of S_n, and since such peaks are lnc points of S, int conv$(p \cup x \cup D)$ must contain points in $\sim S$. Therefore, $x \cup D \not\subseteq \ker S$, and since $D \subseteq \ker S$, this implies that $x \notin \ker S$. We conclude that $\ker S \subseteq D$ and the sets are equal, completing the proof of Theorem 1.

In case D contains a ray but no line, the argument above may be adapted to obtain the following result.

THEOREM 2. *Let D be any closed unbounded convex set in $\mathbf{R}^2$ which contains no line. Then there is a closed set $S \neq D$ in $\mathbf{R}^2$ with* $\ker S = D$.

PROOF. If D has dimension less than 2, a variation of our opening argument in Theorem 1 yields the result. Hence assume that D is fully 2-dimensional. For convenience, orient D so that int D contains the negative y axis, D does not contain either the positive or the negative x axis, and D has a horizontal line of support at the origin. For point $x \notin D$, there are two distinct supporting lines from x to D (which may or may not meet D). The definitions of left and right supporting lines from x to D (in Theorem 1) may be adapted in an obvious manner to this case. When such a supporting line meets D, we may talk about the corresponding left or right supporting segment from x to D.

We proceed as follows to define set S. In case D is a cone, D may be represented by $\bigcup\{R(v, c): c \in C\}$ where C is some compact convex set not containing point v. Then it is easy to construct S by creating an lnc point appropriately at v. For the remainder of the argument, we will assume that D is not a cone. Select points z_L and z_R in bdry D, z_L in the left open halfplane, z_R in the right. Also, since D is not a cone, we may choose these points so that at least one of the segments $[z_L, \theta]$ and $[z_R, \theta]$ is not in bdry D. Let T_L and T_R denote lines which support D at z_L and z_R, respectively, and let A denote the open convex region determined by T_L and T_R and containing int D. Either A will be the interior of a cone whose vertex lies in the upper halfplane or A will be an open parallel strip.

Define B to be the bounded component of (bdry D) $\sim (T_L \cup T_R)$. By our choice of z_L and z_R, it is clear that B is not empty and cl B contains the origin. Furthermore, if F denotes the component of $A \sim D$ whose boundary contains B, then for x in F, each supporting segment from x to D necessarily meets cl B. Select point p_1 in F and define $S_1 = \text{conv}(p_1 \cup D)$. Using p_1, T_L, and T_R defined above, employ the inductive construction in Theorem 1. At each stage of the construction, every peak p_{ki} will lie in the region F. With minor modifications in terminology, the previous argument may be duplicated to define the closed set S and to verify that $\ker S = D$. This completes the proof of Theorem 2.

If the closed set D contains a line, then no appropriate set S exists, as our final theorem reveals.

THEOREM 3. *Let D be any convex set in $\mathbf{R}^2$ which contains a line L. Then D is the kernel of some planar set $S \neq D$ if and only if there is a line M in* bdry D *with* $M \cap D = \varnothing$.

PROOF. We dispose of the easy case first. If bdry D contains such a line M, select distinct points p and q in M. Defining $S = D \cup \{p, q\}$, standard arguments reveal that $D = \ker S$.

To establish the converse statement, assume that no such line M exists. If $D = \mathbf{R}^2$, the result is immediate, so suppose that this is not the case. Then clearly bdry $D = H \cup J$ where H and J are lines parallel to L (and not necessarily distinct). Furthermore, by hypothesis, $H \cap D \neq \varnothing$ and $J \cap D \neq \varnothing$. Let S denote any planar set distinct from D with $D \subseteq \ker S$. (Clearly $\mathbf{R}^2$ itself is such a set.) To finish the proof of Theorem 3, it suffices to show that $\ker S \nsubseteq D$.

In case $S \subseteq \operatorname{cl} D$, then since $H \cap D \neq \varnothing$ and $J \cap D \neq \varnothing$, it is easy to show that S is convex. Thus $\ker S = S \nsubseteq D$, and the theorem is proved. Therefore, throughout the remainder of the argument we may restrict our attention to the case in which $S \nsubseteq \operatorname{cl} D$.

The following notation will be useful. For each point z in $S \sim L$, let M_z denote the line through z and parallel to L, with P_z the open parallal strip bounded by M_Z and L. Notice that since $L \subseteq D \subseteq \ker S$, $P_z \subseteq S$.

To show that $\ker S \nsubseteq D$, select a point x in $S \sim \operatorname{cl} D \neq \varnothing$. There is a neighborhood N of x disjoint from $\operatorname{cl} D$, and we may choose some point x' in $N \cap P_x \subseteq S \sim \operatorname{cl} D$. We assert that $x' \in \ker S$: Choose $y \in S$ to show that $[x', y] \subseteq S$. If $y \in L \subseteq \ker S$, then it is immediate that $[x', y] \subseteq S$, so we assume that $y \in S \sim L$. Then $P_x \cup P_y \cup L$ is a (not necessarily open) parallel strip in S with x' in its interior and y in its closure. Thus $[x', y) \subseteq P_x \cup P_y \cup L \subseteq S$ and $[x', y] \subseteq S$. We conclude that $x' \in \ker S \sim \operatorname{cl} D \subseteq \ker S \sim D$ and $\ker S \nsubseteq D$. The proof of Theorem 3 is established.

We conclude the planar case with two corollaries. The first may be proved using an argument from Theorem 3 above.

COROLLARY 1. *Let D be any convex set in $\mathbf{R}^2$ which contains a line. Then D is not the kernel of any planar set which property contains* cl D.

The second corollary, our characterization theorem, is an immediate consequence of Theorems 1, 2, and 3.

COROLLARY 2. *Let D be a nonempty closed convex subset of $\mathbf{R}^2$. Then D is the kernel of some planar set $S \neq D$ if and only if D contains no line.*

Notice that Corollary 2 above fails without the requirement that D be closed, as Theorem 3 reveals.

3. The results in $\mathbf{R}^d$. Theorem 1 may be modified to obtain the following extension from $\mathbf{R}^2$ to $\mathbf{R}^d$.

THEOREM 4. *Let D be a nonempty compact set in $\mathbf{R}^d$, $d \geqslant 2$. Then there is a compact set $S \neq D$ in $\mathbf{R}^d$ with* $\ker S = D$.

PROOF. The inductive construction for S is outlined below. We restrict our attention to $\mathbf{R}^d$ for $d \geqslant 3$, and assume that D is fully d-dimensional. Consider the family $\mathcal{F}$ of closed halfspaces of the form $[\alpha : a] \equiv \{z\colon z \in \mathbf{R}^d \text{ and } \alpha \cdot z \geqslant a\}$,

where α is any d-dimensional vector having rational coordinates and where a is any rational scalar. It is known that $\mathcal{F}$ is an intersectional basis for the set of compact convex subsets of $\mathbf{R}^d$. Furthermore, if we define $\mathcal{H} \equiv \{F\colon F \text{ in } \mathcal{F} \text{ and } D \subseteq F\}$, then $D = \bigcap \{F\colon F \text{ in } \mathcal{H}\}$. Members of $\mathcal{H}$ may be indexed $H_1, H_2, \ldots$. For convenience of notation, we will let J_i denote the supporting hyperplane for H_i, $1 \leqslant i < \infty$.

To begin the induction, select the first hyperplane J_1 and choose a point x_1 in $J_1 \sim D \neq \varnothing$. Define $S_1 = \operatorname{conv}(x_1 \cup D)$. Choose a real number f_1 so that $0 < f_1 \leqslant \frac{1}{2}\operatorname{dist}(x_1, D)$, and let $T_1 = \operatorname{conv}(x_1 \cup N(D, f_1))$, where $N(D, f_1) = \{z\colon \operatorname{dist}(z, D) \leqslant f_1\}$.

Inductively, assume that for $k \geqslant 2$, $x_1, \ldots, x_k$, $S_1, \ldots, S_k$, $f_1, \ldots, f_k$ and $T_1, \ldots, T_k$ are defined, where

$$S_k = \bigcup_{1 \leqslant i \leqslant k} \{\operatorname{conv}(x_i \cup D)\},$$

$$0 < f_k \leqslant \tfrac{1}{2}\min\{\operatorname{dist}(x_i, \operatorname{conv}(x_j \cup D))\colon i \neq j,\ 1 \leqslant i, j \leqslant k\},$$

$$T_k = \bigcup_{1 \leqslant i \leqslant k} \{\operatorname{conv}(x_i \cup N(D, f_k))\},$$

and

$$x_i \notin \operatorname{conv}(x_j \cup N(D, f_k)) \quad \text{for } i \neq j,\ 1 \leqslant i, j \leqslant k.$$

We would like for hyperplane J_{k+1} to meet int $N(D, f_k)$. If this does not occur, then by our choice of $\mathcal{H}$, there must be a halfspace H_n in $\mathcal{H}$, $n > k + 1$, for which the corresponding hyperplane J_n is parallel to J_{k+1} and for which $J_n \cap \operatorname{int} N(D, f_k) \neq \varnothing$. Note that $H_n \subseteq H_{k+1}$. Remove J_{k+1} and H_{k+1} from our list and replace them with J_n and H_n, respectively. Then renumber appropriate J and H sets so that their indices are consecutive. (That is, J_n will be renamed J_{k+1} and J_{n+m} will be renamed J_{n+m-1} for $m \geqslant 1$. Corresponding H sets will be renamed, too.)

Now $J_{k+1} \cap \operatorname{int} N(D, f_k) \neq \varnothing$. Since $d \geqslant 3$, we may prove that $J_{k+1} \cap T_k \sim S_k \neq \varnothing$, and select point $x_{k+1} \in J_{k+1} \cap T_k \sim S_k$. Clearly $x_{k+1} \notin \operatorname{conv}(x_i \cup D)$ for $1 \leqslant i \leqslant k$. Show that $x_i \notin \operatorname{conv}(x_{k+1} \cup D)$ for $1 \leqslant i \leqslant k$, to conclude that $x_i \notin \operatorname{conv}(x_j \cup D)$ for $i \neq j$, $1 \leqslant i, j \leqslant k + 1$.

Define $S_{k+1} = \bigcup_{1 \leqslant i \leqslant k+1} \{\operatorname{conv}(x_i \cup D)\}$. Choose real number f_{k+1} so that $0 < f_{k+1} \leqslant \frac{1}{2}\min\{\operatorname{dist}(x_i, \operatorname{conv}(x_j \cup D))\colon i \neq j,\ 1 \leqslant i, j \leqslant k + 1\}$, and let $T_{k+1} = \bigcup_{1 \leqslant i \leqslant k+1}\{\operatorname{conv}(x_i \cup N(D, f_{k+1}))\}$. Notice that by our choice of f_{k+1}, $x_i \notin \operatorname{conv}(x_j \cup N(D, f_{k+1}))$ for $i \neq j$, $1 \leqslant i, j \leqslant k + 1$. Observe that $T_{k+1} \subseteq T_k$ and that $\bigcup_{1 \leqslant i \leqslant k+1} S_i \subseteq \bigcap_{1 \leqslant i \leqslant k+1} T_i$. By induction, S_n is defined for every $1 \leqslant n$, and we let $S = \operatorname{cl}(\bigcup_{1 \leqslant n} S_n)$. Using the facts that $\{f_n\}$ converges to zero and $x_i \notin \operatorname{conv}(x_j \cup N(D, f_n))$ for $1 \leqslant j \leqslant n$, $j \neq i$, one may show that each x_i selected above sees via some T_n (and hence via S) only points in the corresponding closed halfspace H_i. Then it is not hard to prove that set S satisfies the theorem.

We close with an easy analogue of Theorem 3.

THEOREM 5. *Let D be any convex set in $\mathbf{R}^d$ which contains a hyperplane. Then D is the kernel of some set $S \neq D$ in $\mathbf{R}^d$ if and only if there is a line L in* bdry *D with $L \cap D = \varnothing$.*

COROLLARY 1. *Let D be a closed convex subset of $\mathbf{R}^d$ which contains a hyperplane. Then D is not the kernel of any set $S \neq D$ in $\mathbf{R}^d$.*

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