AN INFINITE FAMILY IN $\pi_* S^0$ DERIVED FROM MAHOWALD'S $\eta_j$ FAMILY

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Abstract. Combining the relationship due to D. S. Kahn between $\cup_i$ operations in homotopy and Steenrod operations in the $E_2$ term of the Adams spectral sequence with Mahowald's result that $h_i h_j$ is a permanent cycle for $j > 4$, we show that $h_i h_j^2$ is also a permanent cycle for $j > 5$. This gives another infinite family of nonzero elements in the stable homotopy of spheres. Properties of the $\cup_i$ homotopy operations further imply that these elements generate $Z_2$ direct summands.

Our objective is to prove the following theorem.

Theorem. For $j > 5$, $h_i h_j^2$ is a permanent cycle in the mod 2 Adams spectral sequence of $S^0$. It detects a $Z_2$ direct summand of $\pi_* S^0$, $n = 1 + 2^j + 1$.

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Starting from Mahowald's result, that $h_i h_j$ is a permanent cycle for all $j > 4$ [5], the proof is an easy application of the $\cup_i$ homotopy operations. We begin by defining them.

Let $D_2 X$ be the quadratic construction on $X$. That is, if $X$ is a space and $(\ )^+$ denotes addition of a disjoint basepoint $+$, then $D_2 X = ((S^\infty)^+ \wedge X \wedge X)/Z_2$, where $Z_2$ acts by sending $(r, x_1, x_2)$ to $(-r, x_2, x_1)$ and $(+, x_1, x_2)$ to $(+, x_2, x_1)$. If $X$ is a spectrum then the construction of $D_2 X$ is more complicated. Details will be given in [2]. If $\Sigma^\infty X$ is the suspension spectrum of a space $X$ then we have a natural isomorphism $D_2 \Sigma^\infty X \simeq \Sigma^\infty D_2 X$ [2]. We will write as if we were using the spectrum construction for convenience (referring to $\pi_\ast S^0$ rather than $\pi_{i+n} S^n$, $n$ large, for example), but the space level results of [3] suffice.

If $a \in \pi_m D_2 S^n$ then $a$ induces a homotopy operation $a^\ast: \pi_m S^0 \to \pi_m S^0$ (where $S^i$ is the $i$-sphere spectrum) as follows. For $x \in \pi_n S^0$, we let $a^\ast(x)$ be the composite

$$S^m \xrightarrow{\alpha} D_2 S^m \xrightarrow{D_2 x} D_2 S^0 \xrightarrow{\xi} S^0$$

where $\xi = \Sigma^\infty \xi_1: D_2 S^0 \simeq \Sigma^\infty (BZ_2)^+ \to \Sigma^\infty (S^0) = S^0$ is the map of spectra induced by the unique nontrivial map of based spaces $\xi_1: (BZ_2)^+ \to S^0$.

We point out in passing that $a^\ast$ is not a homomorphism. In fact, $a^\ast(x + y) = a^\ast(x) + \tau(a)xy + a^\ast(y)$ where $\tau: D_2 S^n \to S^{2n}$ is a spectrum level transfer map.

The theory of these homotopy operations will be developed in [2].
It is well known that $D_2S^n = \Sigma^P_n$ where $P_n = RP^\infty/RP^{n-1}$ [3]. Let us also write $P^{n+i}_n = RP^{n+i}/RP^{n-1}$. Obstruction theory implies that if $S^0 = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ is an Adams resolution for ordinary mod 2 homology, and if $x \in \pi_nS^0$ is represented by a map $S^n \to X$, then $\xi D_2x$ induces a commutative diagram

$$
\begin{array}{c}
D_2S^n = \Sigma^P_n \supset \Sigma^P_n+1 \supset \cdots \supset \Sigma^P_{n+i} \supset \cdots \supset \Sigma^P_{n+i-1} \supset \Sigma^P_n = S^{2n} \\
\downarrow \quad \downarrow \quad \cdots \quad \downarrow \quad \cdots \quad \downarrow \\
S^0 = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_{2s-1} \leftarrow X_{2s}
\end{array}
$$

[3, Proposition 4.2]. These maps send the characteristic maps of the top cells of each $\Sigma^P_{n+i}$, $c_i \in \pi_{n+i}(\Sigma^P_{n+i}, \Sigma^P_{n+i-1})$, to familiar algebraic constructions on the representative of $x$, $\tilde{x} \in E_2^{n+i} = Ext^A_{n+i}(\mathbb{Z}_2, \mathbb{Z}_2)$, where $A$ is the mod 2 Steenrod algebra. Precisely, we have the following theorem.

**Theorem** [3, Theorem 4.4]. The image of $c_i$ in $E_2 = Ext^{2s-i,2n+2s}$ is $\tilde{x} \cup_i \tilde{x}$.

Several different notations have been used for $\tilde{x} \cup_i \tilde{x}$. We prefer to write $Sq_i\tilde{x}$ for $\tilde{x} \cup_i \tilde{x}$ and reserve $\cup_i$ for use in homotopy. The squaring operations here are those which apply to the cohomology $Ext_A(M, N)$ of comodules $M$ and $N$ over a commutative Hopf algebra $A$ (or, dually, modules $M$ and $N$ over a cocommutative Hopf algebra $A$) [4, §5], [6, §11]. In particular, if $\tilde{x} \in Ext^{n+s}$ then there are elements $Sq_i\tilde{x} \in Ext^{2s-2i,2n+2s}$ for $0 < i < s$, and $Sq_0\tilde{x} = \tilde{x}^2$.

It is apparent then that the differentials on $Sq_i\tilde{x}$ are the successive lifts of the composite $S^{2n+i-1} \to \Sigma^P_{n+i-1} \to X_{2s-i}$ of the $i$-fold suspension of the attaching map of the $n + i$ cell of $P_n$ and the map induced by $\xi D_2x$. In particular, when the attaching map is nullhomotopic, $Sq_i\tilde{x}$ is a permanent cycle. In addition, $c_i$ can then be lifted to an element of $\pi_{n+i}(\Sigma^P_{n+i}, \Sigma^P_{n+i-1})$ which defines a homotopy operation that we call $\cup_i: \pi_n \to \pi_{2n+i}$. Clearly $\cup_i(x)$ is detected by $Sq_i\tilde{x}$.

We are now ready to prove the theorem. Let $x$ be Mahowald’s $\eta_j$, detected by $h_1h_j$. Then $s = 2$ and $n = 2j$. Computing Steenrod operations in $H^*P_n$ shows that $\Sigma^P_n+2 = S^n \vee (S^{2n+1} \cup_2 e^{2n+2})$. Thus $\cup_0(\eta_j) = \eta_j^2$ and $\cup_1(\eta_j)$ are defined but $\cup_2(\eta_j)$ is not. The attaching map of the $2n + 2$ cell shows that $2 \cup_1(\eta) = 0$. The corresponding elements in the mod 2 Adams spectral sequence are

$$
\begin{align*}
Sq_0(h_1h_j) &= h_1^2h_j^2, \\
Sq_1(h_1h_j) &= h_1^3h_{j+1} + h_2h_j^2, \quad \text{and} \\
Sq_2(h_1h_j) &= h_2h_{j+1}.
\end{align*}
$$

This is immediate from the Cartan formula and the formulas $Sq_0(h_j) = h_j^2$ and $Sq_1(h_j) = h_{j+1}$ [1, p. 36 and Theorem 2.5.1]. Therefore $h_1^2h_j^2$ and $h_2h_{j+1} + h_2h_j^2$ are permanent cycles while $d_2(h_2h_{j+1}) = h_0h_2h_j^2$. (This differential is also immediate from the Hopf invariant one differential $d_2(h_jh_{j+1}) = h_0h_2h_j^2$. The Hopf invariant one differential is in turn an immediate consequence of the above formulas and the fact that if $m$ is odd then $P_m^{n+1} = S^n \cup_2 e^{m+1}$.) Since $h_1^2h_{j+1}$ is a permanent cycle detecting $\eta_{j+1}$, $h_2h_j^2$ is a permanent cycle detecting $\eta_j = \cup_1(\eta_j) = \eta_{j+1}$. It is known that $h_2h_j^2 \neq 0$ if $j > 4$ [1, Theorem 2.5.1]. Also $h_2h_j^2$ is not a boundary since
there are no elements which can hit it. Thus $\tau_j$ is nonzero. Since $2 \cup_i(\eta_j) = 0$ and $2 \eta = 0$, $\tau_j$ has order 2. Since there are no elements of lower filtration in the $2n + 1 = 1 + 2^{i+1}$ stem, $\tau_j$ is not divisible by 2. It follows that $\tau_j$ generates a $\mathbb{Z}_2$ direct summand of the $1 + 2^{i+1}$ stem.

Note that the differential $d_2(h_2 h_{j+1}) = h_0 h_2 h_2$ is, as usual, not sufficient to imply that $2\tau_j = 0$. For this, the factorization of $\cup_i(\eta_j)$ through $\Sigma^n p_{n+1} = S^{2n+1} \cup_2 e^{2n+2}$ is needed.

References


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