PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, August 1981

COMPLETELY REGULAR AND ω-REGULAR SPACES

D. C. KENT AND G. D. RICHARDSON[1]

Dedicated to the memory of Professor Shouro Kasahara

ABSTRACT. The completely regular convergence spaces are characterized as those spaces having symmetric compactifications. The ω-regular convergence spaces are those which have regular compactifications.

1. Introduction. It has never been obvious how "complete" regularity should be defined for convergence spaces. If one equates "completely regular" with "uniformizable" (in the sense of being compatible with a uniform convergence structure), the concept is too weak to be useful. In **[3]**, A. C. Cochran and R. B. Trail suggested a definition which, essentially, makes complete regularity of a convergence space equivalent to complete regularity of the topological modification. The approach that we shall take is based upon the fact that a topological space is completely regular iff it is the subspace of a compact regular topological space. However, as we shall see, this statement cannot be translated verbatim into the convergence space setting.

In **[8]**, the authors showed that a convergence space X has a regular, Hausdorff compactification iff X is regular and has the same ultrafilter convergence as a Tychonoff topological space. We shall extend the definition of complete regularity given in **[8]** for Hausdorff spaces to arbitrary spaces by defining a convergence space X to be *completely regular* if X is symmetric (defined later) and has the same ultrafilter convergence as a completely regular topological space. This definition yields a much stronger notion of complete regularity than that suggested in **[3]**.

It would be reasonable to conjecture that, as in the topological case, a space is completely regular iff it is the subspace of a compact regular space. On the contrary, we shall show that the class of spaces having regular compactifications is the larger class of ω-regular spaces, an important class which includes the c-embedded spaces of E. Binz **[1]** in addition to the completely regular spaces.

We do, however, obtain a similar characterization for the completely regular spaces. These turn out to be precisely the class of spaces which have symmetric

---

Received by the editors June 10, 1980; presented at the Conference on Convergence Spaces, Cameron University, Lawton, Oklahoma, April 1980.

1980 *Mathematics Subject Classification.* Primary 54A20; Secondary 54D30, 54D35, 54C25.

Key words and phrases. Convergence space, completely regular space, ω-regular space, regular compactification, symmetric compactification.

[1]The second author wishes to thank the East Carolina University Research Committee for partial financial support.

compactifications. The symmetric spaces form a subclass of the class of regular spaces which includes the regular topological spaces. This characterization enables one to define completely regular spaces entirely in terms of convergence space criteria.

2. Completely regular spaces. For definitions, notation, and terminology concerning convergence spaces, the reader is referred to [4] and [5]. A space X is said to be *symmetric* if X is regular and $\mathcal{F} \to y$ whenever $\mathcal{F} \to x$ and $\dot{x} \to y$. Our first proposition is essentially Theorem 2.4 of [5].

PROPOSITION 2.1. (a) *If X is a compact regular space, then the second iteration of the closure operator of X is idempotent.*

(b) *If X is a compact symmetric space, then X has the same ultrafilter convergence as a compact regular topological space.*

Given a space X, we shall denote by λX (respectively, ωX, σX, X_r) the topological (resp., completely regular, symmetric, regular) modification of X. In other words, λX is the finest topological space on $|X|$ (the underlying set of X) coarser than X; the other modifications can be similarly characterized.

PROPOSITION 2.2. *Let f: $X \to Y$ be a continuous function. In the following commutative diagram, where the vertical arrows are the function f and the horizontal arrows are identity functions, all functions are continuous.*

$$\begin{array}{ccccccc} X & \to & X_r & \to & \sigma X & \to & \omega X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \to & Y_r & \to & \sigma Y & \to & \omega Y \end{array} \qquad \begin{array}{ccc} X & \to & \lambda X \\ \downarrow & & \downarrow \\ Y & \to & \lambda Y \end{array}$$

A concept which plays a key role in the proof of our two main theorems is the compactification (X^*, j) of an arbitrary space X (see [7]). The details of the construction of X^* are not essential for what follows, but it is useful to know that X is a subspace of X^*, so that j: $X \to X^*$ is the identity embedding. The crucial result that is needed is stated in the following proposition.

PROPOSITION 2.3. *If f: $X \to Y$ is continuous and Y is compact and regular, then there is a continuous function f^*: $X^* \to Y$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ j\downarrow & \nearrow f^* & \\ X^* & & \end{array}$$

The proof of Proposition 2.3 is given in [7] in the case that X and Y are Hausdorff. The generalization of the proof to the non-Hausdorff case is easy, although f^* is no longer unique in the general case. The symbols σX^*, ωX^*, and X_r^* will be used to represent the respective modifications of X^*.

THEOREM 2.4. *A convergence space is completely regular iff it has a symmetric compactification.*

PROOF. Suppose that X is a completely regular space. Since $X = \sigma X$, it follows by Proposition 2.2 that j: $X \to \sigma X^*$ is a continuous map. If $j(\mathcal{F}) \to j(x)$ in σX^*, then there is $\tau \in X^*$ such that $\dot{\tau} \to j(x)$ and $j(\mathcal{F}) \to \tau$ in X_r^* (see [5, p. 230]). Hence (see [4, p. 25]) there is a filter $\mathcal{G}$ on X and a natural number n such that $j(\mathcal{F}) \geq \mathrm{cl}^n_{X_r^*} j(\mathcal{G})$, where $j(\mathcal{G}) \to \tau$ in X^*. If $\tau \notin j(X)$, then, by construction of X^*, $\mathcal{G}$ may be assumed to be an ultrafilter. Since X is completely regular and $\mathcal{G} \not\to x$ in X, there is a continuous function f: $X \to [0, 1]$ such that $f(\mathcal{G}) \not\to f(x)$ in $[0, 1]$. If f^*: $X_r^* \to [0, 1]$ is the continuous extension of f described in Proposition 2.3, then $f(\mathcal{G}) = f^*(j(\mathcal{G})) \to f^*(\tau) = f(x)$ in $[0, 1]$, a contradiction. Hence $\tau = j(z)$ for some $z \in X$. The same reasoning shows that $\dot{z} \to x$ in X.

It follows that $\mathcal{F} \geq j^{-1}(\mathrm{cl}^n_{X_r^*} j(\mathcal{G})) \geq j^{-1}(\mathrm{cl}_{\omega X^*} j(\mathcal{G})) = \mathrm{cl}_{\omega X}\mathcal{G} = \mathrm{cl}_X \mathcal{G}$, since X is completely regular. Since $j(\mathcal{G}) \to \tau$ in X^*, $\mathcal{G} \to z = j^{-1}(\tau)$ in X. Thus $\mathrm{cl}_X\mathcal{G} \to z$, which implies $\mathcal{F} \to z$ in X. Since X is symmetric and $\dot{z} \to x$ in X, $\mathcal{F} \to x$ in X. This implies that j: $X \to \sigma X^*$ is a dense embedding, and therefore $(\sigma X^*, j)$ is a symmetric compactification of X.

The converse follows directly from Proposition 2.1(b). □ It should be mentioned that, from Proposition 2.3, a continuous map from X into a compact symmetric space has a continuous extension to σX^*. The preceding result leads to an alternate characterization of the completely regular modification.

PROPOSITION 2.5. *For any space X, ωX is the restriction of σX^* to $|X|$.*

PROOF. Let X' be the restriction of σX^* to $|X|$; then X' is completely regular and $X' \leq X$. If X'' is any completely regular space such that $|X''| = |X|$ and the identity map i: $X \to X''$ is continuous (i.e., $X'' \leq X$), then by Proposition 2.3 $j'' \circ i$ has a continuous extension i^*: $X^* \to \sigma(X'')^*$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{i} & X'' \\ j\downarrow & & \downarrow j'' \\ \sigma X^* & \xrightarrow{i^*} & \sigma(X'')^* \end{array}$$

Since $i^*|X = i$, and X'' is a subspace of $\sigma(X'')^*$ by Theorem 2.4 it follows that i: $X' \to X''$ is continuous. Thus $X' = \omega X$. □

3. ω-regular spaces. A space is defined to be ω-regular if $\mathrm{cl}_{\omega X}\mathcal{F} \to x$ whenever $\mathcal{F} \to x$. It is shown in [6] that a space X is c-embedded (see [1], [2]) iff X is ω-regular and pseudotopological and has enough continuous real-valued functions to separate points.

PROPOSITION 3.1. *If X is a compact regular space, then X is ω-regular and $\mathrm{cl}_{\omega X} A = \mathrm{cl}^2_X A$ for each subset A of X.*

PROOF. Let $A \subseteq X$. By Proposition 2.1(b), $\lambda\sigma X = \lambda\omega X$ is a completely regular topological space which has the same ultrafilter convergence as σX. By Proposition 2.6(1) of [5], $\mathrm{cl}_{\sigma X} A \subseteq \mathrm{cl}^2_X A$. These facts imply that $\mathrm{cl}_{\omega X} A = \mathrm{cl}_{\sigma X} A = \mathrm{cl}_{\lambda X} A = \mathrm{cl}^2_X A$. □

THEOREM 3.2. *A convergence space X is ω-regular iff it has a regular compactification.*

PROOF. Let X be an ω-regular space. Then $j: X \to X_r^*$ is continuous by Proposition 2.2, and by Proposition 3.1 X_r^* is an ω-regular convergence space. If $j(\mathcal{F}) \to j(x)$ in X_r^*, then there are $\mathcal{G} \to x$ in X and a natural number n such that $j(\mathcal{F}) \geq \text{cl}^n_{X_r^*} j(\mathcal{G})$. Thus $\mathcal{F} \geq j^{-1}(\text{cl}^n_{X_r^*} j(\mathcal{G})) \geq \text{cl}_{\omega X} \mathcal{G}$. The latter filter converges to x in X since X is ω-regular. Therefore (X_r^*, j) is a regular compactification of X.

Conversely, if X has a regular compactification, then X is a subspace of an ω-regular space, and therefore ω-regular. □

Moreover, a continuous map from X into a compact regular space has a continuous extension to X_r^*. Let X be any space, and let ρX be the subspace of X_r^* determined by $|X|$. One can show, as in the proof of Proposition 2.5, that ρX is the finest ω-regular space coarser than X; that is, ρX is the ω-regular modification of X. For any space X, $\omega X \leq \rho X \leq X_r \leq X$.

Examples are given in [2] and [6] of Hausdorff topological spaces which are ω-regular but not completely regular. Furthermore, examples abound of regular spaces which are not ω-regular.

Despite the similarity of their characterizations, completely regular and ω-regular spaces are quite different concepts. Although every ω-regular space is a subspace of a compact regular space, a space for which the second iteration of the closure operator is idempotent, an example in [9] shows that an ω-regular space can have infinitely many distinct iterations of its closure operator. This shows that ω-regular spaces can be quite "nontopological", whereas the completely regular spaces are barely distinguishable from their topological counterparts.

REFERENCES

1. E. Binz, *Continuous convergence on $C(X)$*, Lecture Notes in Math., vol. 469, Springer-Verlag, Berlin, Heidelberg and New York, 1975.

2. H. P. Butzmann and B. Müller, *Topological c-embedded spaces*, General Topology Appl. **6** (1976), 17–20.

3. A. C. Cochran and R. B. Trail, *Regularity and complete regularity for convergence spaces*, Proc. Virginia Polytechnic Institute Topology Conf., Lecture Notes in Math., vol. 375, Springer-Verlag, Berlin and New York, 1973, pp. 64–70.

4. D. C. Kent and G. D. Richardson, *The regularity series of a convergence space*, Bull. Austral. Math. Soc. **13** (1975), 21–44.

5. ______, *The regularity series of a convergence space*. II, Bull. Austral. Math. Soc. **15** (1976), 223–244.

6. D. Kent, K. McKennon, G. Richardson and M. Schroder, *Continuous convergence in $C(X)$*, Pacific J. Math. **52** (1974), 271–279.

7. G. D. Richardson, *A Stone-Čech compactification for limit spaces*, Proc. Amer. Math. Soc. **25** (1970), 403–404.

8. G. D. Richardson and D. C. Kent, *Regular compactification of convergence spaces*, Proc. Amer. Math. Soc. **31** (1972), 571–573.

9. ______, *A note on $C_c(X)$*, Proc. Amer. Math. Soc. **49** (1975), 441–445.

DEPARTMENT OF MATHEMATICS, WASHINGTON STATE UNIVERSITY, PULLMAN, WASHINGTON 99164

DEPARTMENT OF MATHEMATICS, EAST CAROLINA UNIVERSITY, GREENVILLE, NORTH CAROLINA 27834