PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, August 1981

SHORTER NOTES

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TOTALLY REAL KLEIN BOTTLES IN $\mathbf{C}^2$

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ABSTRACT. It is proved that $\mathbf{C}^2$ contains totally real compact submanifolds that are nonorientable, being homeomorphic to a Klein bottle.

A smooth manifold M in $\mathbf{C}^n$ is said to be *totally real* if none of its tangent spaces $T_p(M)$ (where $p \in M$) contains a complex subspace. In other words, $T_p(M)$ and $iT_p(M)$ are to have only 0 in common. When the (real) dimension of M is n (briefly, when M is an n-manifold) this amounts to requiring that the $\mathbf{C}$-span of every $T_p(M)$ is all of $\mathbf{C}^n$.

The totally real submanifolds of $\mathbf{C}^n$ play an important role in problems about approximation of continuous functions by holomorphic ones. See **[3]**, where further references are given, and also **[1]**, where these manifolds occur in a different context.

The most obvious totally real n-manifold in $\mathbf{C}^n$ is R^n, the set of all points in $\mathbf{C}^n$ whose coordinates are real. As far as compact manifolds are concerned, Wells **[2]** has proved that the Euler characteristic of every compact *orientable* totally real n-manifold in $\mathbf{C}^n$ is zero. Among the compact orientable 2-manifolds there is therefore only one, namely the torus, that has totally real embeddings in $\mathbf{C}^2$. The set of all points $(e^{i\theta}, e^{i\varphi})$, where $(\theta, \varphi) \in R^2$, is the standard example of such an embedding of the torus T^2.

However, $\mathbf{C}^2$ also contains compact totally real 2-manifolds that are *not* orientable, a fact that has apparently escaped earlier notice.

THEOREM. *There is an entire map F: $\mathbf{C}^2 \to \mathbf{C}^2$ such that $F(R^2)$ is totally real and homeomorphic to a Klein bottle.*

PROOF. Pick constants $a, b,\ a > b > 0$, put

$$g(\theta, \varphi) = (a + b \cos \varphi)e^{i\theta}, \qquad h(\varphi) = \sin \varphi + i \sin 2\varphi,$$

Received by the editors November 21, 1980.

1980 *Mathematics Subject Classification.* Primary 32E30; Secondary 57R40.

[1]This research was partially supported by NSF Grant MCS 78-06860 and by the William F. Vilas Trust Estate.

and define $F(\theta, \varphi) = (z, w)$ by

$$z = g^2(\theta, \varphi), \qquad w = g(\theta, \varphi)h(\varphi).$$

Then $F(\theta, \varphi + 2\pi) = F(\theta, \varphi) = F(\theta + \pi, -\varphi)$, for all $(\theta, \varphi) \in \mathbf{C}^2$. To prove that F separates all other pairs of points in the region

$$Q = \{(\theta, \varphi): -\pi \leq \theta < \pi, -\pi \leq \varphi < \pi\} \subset R^2,$$

pick $(z, w) \in F(Q)$ and note that

$$a + b \cos \varphi = |z|^{1/2},$$

so that z determines $\cos \varphi$ and $e^{2i\theta}$. This gives two choices for $e^{i\theta}$. Once one of these is made, $\sin \varphi$ is determined by w.

We conclude that F is 2-to-1 on Q, and that $K = F(R^2)$ is a Klein bottle. [Note the minus sign in $F(\theta, \varphi) = F(\theta + \pi, -\varphi)$.]

For each $(\theta, \varphi) \in R^2$, the column vectors

$$\begin{pmatrix} z_\theta \\ w_\theta \end{pmatrix} = \begin{pmatrix} 2gg_\theta \\ hg_\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_\varphi \\ w_\varphi \end{pmatrix} = \begin{pmatrix} 2gg_\varphi \\ hg_\varphi + gh' \end{pmatrix}$$

are tangent to K at $F(\theta, \varphi)$. They are linearly independent (over $\mathbf{C}$) since

$$z_\theta w_\varphi - z_\varphi w_\theta = 2g^2 g_\theta h' = 2ig^3 h' \neq 0.$$

[Note that $h'(\varphi) \neq 0$ for all real φ.] The $\mathbf{C}$-span of these tangent vectors is therefore all of $\mathbf{C}^2$, so that K is totally real.

References

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3. ______, *Real analytic subvarieties and holomorphic approximation*, Math. Ann. **179** (1969), 130–141.

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