A COUNTEREXAMPLE TO THE UNIMODULAR CONJECTURE ON
FINITELY GENERATED DIMENSION GROUPS

NORBERT RIEDEL

Abstract. We give a series of examples of simple finitely generated dimension

groups which cannot be obtained as the inductive limit of a system

\[ Z^\rightarrow Z^\rightarrow \cdots Z^\rightarrow \cdots, \]

where each \( A_n \) is a unimodular matrix whose entries are nonnegative integers.

1. In this note we are concerned with ordered groups \( G \) of the following form. \( G \)
is equal to \( Z^r \) as an abelian group only, for some \( r \in \mathbb{N} \), and there exists a set
\( \{a^{(1)}, \ldots, a^{(p)}\} \) of linearly independent vectors in \( (\mathbb{R}^r)^+ \) such that the positive cone
\( G^+ \) of \( G \) is given by

\[ G^+ = \{ z \in Z^r / \langle a^{(i)}, z \rangle > 0; i = 1, \ldots, p \} \cup \{0\}. \]

\( G \) is a dimension group if it satisfies the Riesz interpolation property (for the
definitions concerning dimension groups we refer to [1], [2]). Effros and Shen
conjectured in [3] (see also [1]) that if \( G \) is a dimension group, then there exists a
sequence \( A_1, A_2, \ldots \) in the set \( \text{GL}(r, \mathbb{Z})^+ \) of all unimodular matrices whose entries
are nonnegative integers, such that

\[ G^+ = \bigcup_{n=1}^{\infty} (A_n \ldots A_1)^{-1}(Z^r)^+. \]

In [4] we have shown that the conjecture is true if \( G \) is simple (i.e. \( G^+ \) has no
nontrivial faces) and \( p = 1 \) holds. Using the theory of diophantine approximation
we will show in the sequel that for \( p = r - 1 \) (\( r > 3 \)) there exist simple dimension
groups for which the conjecture of Effros and Shen is false. If \( p = r - 1 \) holds then
we can use the following criterion in order to decide whether \( G \) is a simple
dimension group or not. Let \( b \) be a nonzero vector which is orthogonal to the
hyperplane in \( \mathbb{R}^r \) which is generated by the vectors \( a^{(1)}, \ldots, a^{(r-1)} \).

1.1. Proposition [2, Corollary 4.2]. The following statements are equivalent.

(1.1.1) \( G \) is a simple dimension group.
(1.1.2) \( \det(a^{(1)}, \ldots, a^{(r-1)}, z) \neq 0 \) holds for any nonzero vector \( z \) in \( Z^r \).
(1.1.3) The components of the vector \( b \) are linearly independent over the rational
field \( \mathbb{Q} \).

Received by the editors October 23, 1980 and, in revised form, January 5, 1981.
1980 Mathematics Subject Classification. Primary 06F20, 10F10, 46L99.
© 1981 American Mathematical Society
0002-9939/81/0000-00402/$02.25

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Henceforth we fix \( r > 3 \). Let \( a^{(1)}, \ldots, a^{(r-1)} \) be fixed linearly independent vectors in \((\mathbb{R}^r)^+\) such that the ordered group \( G \) defined above is a simple dimension group. Denote by \( b \) the unique vector in \( \mathbb{R}^r \) such that

\[
\langle b, x \rangle = \det(a^{(1)}, \ldots, a^{(r-1)}, x).
\]

By multiplying one of the vectors \( a^{(1)}, \ldots, a^{(r-1)} \) with a suitable constant we may assume that the last component of \( b \), equal to 1. Of course, this does not impair the definition of \( G \).

We need some other notations. For any subset \( M \subseteq \mathbb{R}^r \) we denote by \( \text{conv}(M) \) the convex hull of \( M \). For any vector \( x = (x_1, \ldots, x_r)' \in \mathbb{R}^r \setminus \{0\} \) we set \( \bar{x} = \|x\|_1^{-1}x \), where \( \|x\|_1 \) is the \( l^1 \)-norm of \( x \), and we set \( \bar{x} = (x_1, \ldots, x_{r-1})' \).

Now we assume that \( A_1, A_2, \ldots \) is a sequence in \( \text{GL}(r, \mathbb{Z})^+ \) such that

\[
G^+ = \bigcup_{n=1}^{\infty} (A_n \ldots A_1)^{-1}(\mathbb{Z}^r)^+,
\]

or equivalently,

\[
\text{conv}(\{\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}\}) = \bigcap_{n=1}^{\infty} \{ \bar{x} / x \in \text{conv}(A_n^t \ldots A_1^t(\mathbb{Z}^r)^+) \setminus \{0\} \}. \tag{2.1}
\]

**2.1. Definition** [5, II, §4]. To each vector \( x \in \mathbb{R}^{r-1} \) we associate a linear form \( L_x \) on \( \mathbb{Z}^{r-1} \) by \( L_x(z) = \langle x, z \rangle, \) \( z \in \mathbb{Z}^{r-1} \). \( L_x \) is called badly approximate if there exists a positive constant \( \alpha \) such that \( |L_x(z) - q| > \alpha \|z\|_{\infty}^{r+1} \) holds for each \( z \in \mathbb{Z}^{r-1} \setminus \{0\}, \) \( q \in \mathbb{Z} \), where \( \|z\|_{\infty} \) denotes the maximum norm of \( z \).

Our main purpose in this section is to show that in our present situation the linear form \( L_x \), as defined above, can not be badly approximable.

We need the following lemma whose easy proof is left to the reader.

**2.2. Lemma.** Let \( A_1, A_2, \ldots \) be a monotonely decreasing sequence of \( r - 1 \) dimensional simplices in \( \mathbb{R}^r \) and let \( \Delta = \cap_{n=1}^{\infty} \Delta_n \). Moreover let \( v^{(1, n)}, \ldots, v^{(r, n)} \) be the extreme points of \( \Delta_n \). Then there exists a monotonely increasing sequence \( n_1, n_2, \ldots \) of positive integers such that \( \{v^{(j, n_i)}\}_{i \in \mathbb{N}} \) converges to a point in \( \Delta \) for each \( j, 1 < j < r \), and for any extreme point \( w \) in \( \Delta \) there exists a \( j, 1 < j < r \), such that \( w = \lim_{i \to \infty} v^{(j, n_i)} \).

It follows from (2.1) together with 2.2 that there exists a monotonely increasing sequence \( n_1, n_2, \ldots \) of positive integers, a permutation \( \sigma \) of the set of integers \( \{1, \ldots, r\} \), and a point \( w \) in \( \text{conv}(\{\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}\}) \) such that, if \( v^{(6(1)), n_i}, \ldots, v^{(6(r)), n_i} \) are the column vectors of the matrix \( B^{(i)} = A^t_n \ldots A^t_1 \), then we have

\[
\lim_{i \to \infty} v^{(j, n_i)} = \bar{a}^{(j)} \quad \text{for} \quad 1 < j < r - 1; \quad \lim_{i \to \infty} v^{(r, n_i)} = w.
\]

Now we can prove the following proposition which is crucial for the proof of Theorem 2.4.

**2.3. Proposition.** There exists a \( k \in \{1, \ldots, r - 1\} \) such that

\[
\lim_{n \to \infty} \det(v^{(1, n)}, \ldots, v^{(r, n)})^{-1} \det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}), v^{(k, n)}) = 0.
\]
Proof. There exists a \( k \in \{1, \ldots, r-1\} \) such that \( \tilde{a}^{(k)} \neq w \). Since \( \text{conv}(\{\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}\}) \) is a simplex containing \( w \), we have
\[
\tilde{a}^{(k)} \notin M = \text{conv}\{\{\tilde{a}^{(j)}/j \neq k\} \cup \{w\}\}.
\]
It follows that
\[
\delta = \inf\{\|\tilde{a}^{(k)} - x\|/x \in M\} > 0.
\]
Since \( \tilde{a}^{(k)} \) is contained in \( \text{conv}(\{\tilde{v}^{(1,n)}, \ldots, \tilde{v}^{(r,n)}\}) \) for each \( n \in \mathbb{N} \), we can write
\[
\tilde{a}^{(k)} = \sum_{j=1}^{r} t_j^{(n)} \tilde{v}^{(j,n)},
\]
with \( t_j^{(n)} \in [0, 1] \) and \( \sum_{j=1}^{r} t_j^{(n)} = 1 \). By our assumption \( G \) is a simple dimension group. Therefore we obtain from (1.1.2) that \( \det(\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, \tilde{v}^{(k,n)}) \neq 0 \). In particular \( \tilde{v}^{(k,n)} \neq \tilde{a}^{(k)} \) and \( t_k^{(n)} \neq 1 \). Thus we can define
\[
\lambda_j^{(n)} = t_j^{(n)}(1 - t_k^{(n)})^{-1}, \quad u^{(n)} = \sum_{j \neq k} \lambda_j^{(n)} \tilde{v}^{(j,n)}.
\]
Since \( \{\tilde{v}^{(j,n)}\}_{n \in \mathbb{N}} \) converges to \( \tilde{a}^{(j)} \) for \( 1 < j < r-1 \), and to \( w \) for \( j = r \), there exists a \( n_0 \in \mathbb{N} \) such that for any \( n > n_0 \)
\[
\|\tilde{v}^{(j,n)} - \tilde{a}^{(j)}\| < \delta/2 \quad \text{for} \ j \neq r; \quad \|\tilde{v}^{(r,n)} - w\| < \delta/2.
\]
If we set
\[
\tilde{u}^{(n)} = \sum_{j \neq k} \lambda_j^{(n)} \tilde{a}^{(j)} + \lambda_r^{(n)} w
\]
then we obtain for each \( n > n_0 \)
\[
\|u^{(n)} - \tilde{a}^{(k)}\| < \delta/2.
\]
Since \( \tilde{u}^{(n)} \) is contained in \( M \) we have
\[
\|\tilde{u}^{(n)} - \tilde{a}^{(k)}\| \geq \delta.
\]
By combining the last two inequalities we obtain an estimation for the distance of \( u^{(n)} \) and \( \tilde{a}^{(k)} \):
\[
\|u^{(n)} - \tilde{a}^{(k)}\| > \delta/2 \quad \text{for} \ n > n_0.
\]
Since \( \tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, u^{(n)} \) are contained in \( \text{conv}(\{\tilde{v}^{(1,n)}, \ldots, \tilde{v}^{(r,n)}\}) \) we have
\[
|\det(\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, u^{(n)})| < |\det(\tilde{v}^{(1,n)}, \ldots, \tilde{v}^{(r,n)})|.
\]
Therefore we obtain for each \( n > n_0 \)
\[
\left|\frac{|\det(\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, \tilde{v}^{(k,n)})|}{|\det(\tilde{v}^{(1,n)}, \ldots, \tilde{v}^{(r,n)})|}\right|
\leq \left|\frac{|\det(\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, t_k^{(n)-1}(\tilde{a}^{(k)} - u^{(n)}) + u^{(n)})|}{|\det(\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, u^{(n)})|}\right| = |1 - t_k^{(n)}|.
\]
Since \( \|u^{(n)} - \tilde{a}^{(k)}\| > \delta/2 \) holds for each \( n > n_0 \) and since
\[
\|\tilde{a}^{(k)} - \tilde{v}^{(k,n)}\| = |1 - t_k^{(n)}| \cdot \|u^{(n)} - \tilde{a}^{(k)}\|
\]
converges to zero for \( n \to \infty \) it follows that \( |1 - t_k^{(n)}| \) converges to zero. Hence

\[
\lim_{n \to \infty} \det(\tilde{\alpha}^{(1,n)}, \ldots, \tilde{\alpha}^{(r,n)})^{-1} \det(\tilde{\alpha}^{(1)}, \ldots, \tilde{\alpha}^{(r-1)}, \tilde{\alpha}^{(k,n)}) = 0.
\]

2.4. Theorem. In the situation considered above the linear form \( L_2 \) is not badly approximable.

Proof. Suppose that our assertion is not true. Then there exists a positive constant \( \alpha \) such that

\[
|\langle b, z \rangle| > \alpha \|z\|^{-r+1}
\]

for each \( z \in \mathbb{Z}' \setminus \{0\} \). (2.4.1)

First we show the following.

(2.4.2) There exists a positive constant \( \beta \) such that

\[
\|v^{(i,n)}\| \cdot \|v^{(j,n)}\|^{-1} < \beta \quad \text{for each } n \in \mathbb{N}; \quad i, j \in \{1, \ldots, r\}.
\]

Suppose that this is not true. Let \( k_n = \min\{\|v^{(j,n)}\|/1 < j < r\} \) and \( l_n = \|v^{(1,n)}\| \cdot \|v^{(2,n)}\| \cdots \cdot \|v^{(r,n)}\|. \) Then there exists a monotonely increasing sequence \( n_1, n_2, \ldots \) of positive integers such that \( \lim_{i \to \infty} k_n^r l_n^{-1} = 0. \) We can find an \( m \in \{1, \ldots, r\} \) such that \( \|v^{(m,n)}\| = k_n \) for infinitely many \( i \in \mathbb{N}. \) Thus we may choose the sequence \( n_1, n_2, \ldots \) in such a manner that in addition \( \|v^{(m,n)}\| = k_n \) holds for each \( i \in \mathbb{N}. \) We set \( d = \|a^{(1)}\| \cdots \|a^{(r-1)}\|. \) Now we obtain from the inequality

\[
\|v^{(m,n)}\| \cdot \|v^{(m,n)}\|^{-1} < \beta \quad \text{for any } n \in \mathbb{N},
\]

\[
k_n^r l_n^{-1} \det(\tilde{\alpha}^{(1)}, \ldots, \tilde{\alpha}^{(r-1)}, \tilde{\alpha}^{(m,n)}) = k_n^r l_n^{-1} \det(\tilde{\alpha}^{(1)}, \ldots, \tilde{\alpha}^{(r-1)}, \tilde{\alpha}^{(m,n)})
\]

\[
< k_n^r l_n^{-1} \det(\tilde{\alpha}^{(1,n)}, \ldots, \tilde{\alpha}^{(r,n)})
\]

\[
= k_n^r l_n^{-1} d^{-1} |\det(\tilde{\alpha}^{(1,n)}, \ldots, \tilde{\alpha}^{(r,n)})| = k_n^r l_n^{-1} d^{-1}.
\]

From this we infer that

\[
\lim_{i \to \infty} \|v^{(m,n)}\| |\det(\tilde{\alpha}^{(1)}, \ldots, \tilde{\alpha}^{(r-1)}, \tilde{\alpha}^{(m,n)})| = 0.
\]

Since \( \|v^{(m,n)}\| \to \infty < \|v^{(m,n)}\| \) holds for each \( i \in \mathbb{N}, \) this implies

\[
\lim_{i \to \infty} \|v^{(m,n)}\| |\langle b, v^{(m,n)} \rangle| = 0.
\]

However, this contradicts (2.4.1).

Using 2.3 as well as (2.4.2) we can now easily complete the proof of our theorem. Let \( k \in \{1, \ldots, r - 1\} \) be chosen as in 2.3. By (2.4.2) there exists a positive constant \( \beta \) such that

\[
\|v^{(k,n)}\| < \beta \|v^{(1,n)}\| \cdots \|v^{(r,n)}\| \quad \text{for any } n \in \mathbb{N}.
\]

Since \( \det(B_n) = 1 \) holds for each \( n \in \mathbb{N}, \) we obtain now

\[
\beta \|v^{(k,n)}\| \cdot \|v^{(k,n)}\|^{-1} \det(\tilde{\alpha}^{(1,n)}, \ldots, \tilde{\alpha}^{(r,n)})^{-1}
\]

\[
> \|v^{(k,n)}\|^{-1} |\det(\tilde{\alpha}^{(1)}, \ldots, \tilde{\alpha}^{(r-1)}, \tilde{\alpha}^{(k,n)})| \cdot |\det(\tilde{\alpha}^{(1)}, \ldots, \tilde{\alpha}^{(r-1)}, \tilde{\alpha}^{(k,n)})|^{-1}
\]

\[
= \|v^{(k,n)}\|^{-1} d^{-1} |\det(\tilde{\alpha}^{(1)}, \ldots, \tilde{\alpha}^{(r-1)}, \tilde{\alpha}^{(k,n)})|.
\]
Therefore, by 2.3,
\[ \lim_{n \to \infty} \| \delta^{(k,n)} \|_{\infty}^{-1} \langle \delta, v^{(k,n)} \rangle = 0, \]
and this contradicts (2.4.1).

3. By using the results of §2 we are now able to construct a lot of counterexamples to the unimodular conjecture. Let \( x \) be a vector in \( \mathbb{R}^{r-1} \) such that \( L_x \) is badly approximable. By [5, II, §4, Theorem 4A] we may choose \( x \) in such a manner that the components \( x_1, \ldots, x_{r-1} \) of \( x \) lie in an algebraic number field \( K \) of degree \( r \) and \( \{1, x_1, \ldots, x_{r-1}\} \) is a basis of \( K \). We also demand that the components of \( x \) do not all have a positive sign. (If necessary, change the basis in \( \mathbb{Z}^{r-1} \).) We set \( b = (x_1, \ldots, x_{r-1})' \), and we choose \( r - 1 \) linearly independent positive vectors in the hyperplane which is orthogonal to \( b \). (Since the components of \( x \) do not all have a positive sign this can always be done.) Now we define a simple dimension group \( G \) as in §1. In order to show that \( G \) cannot be obtained as the inductive limit of a unimodular system of groups it is necessary to state that the property that \( b \) gives rise to a badly approximable linear form \( L_b \) is left invariant under isomorphisms of the group \( G \). To speak more precisely, if \( G' \) is an isomorphic copy of \( G \) and we associate a vector \( b' \) with \( G' \) in the same manner as the vector \( b \) with \( G \), then we have \( \gamma b' = Ab \) for some \( A \in \text{GL}(r, \mathbb{Z}) \) and a nonzero constant \( \gamma \). Now some simple calculations show that \( L_b \) is also badly approximable. Thus, by an application of 2.4, we can conclude that \( G \) is not isomorphic to the inductive limit of a system \( \mathbb{Z}' \rightarrow \mathbb{Z}' \rightarrow \cdots \mathbb{Z}' \rightarrow \cdots \) with \( A_n \in \text{GL}(r, \mathbb{Z})^+ \) for each \( n \in \mathbb{N} \).

It follows from a theorem of Khintchine (see [5, III, §3, Theorem 3A]) together with a transference principle for badly approximable linear forms (see [5, IV, §5, Theorem 5B]) that the set of all vectors \( x \in \mathbb{R}^{r-1} \) such that \( L_x \) is badly approximable has Lebesgue measure zero. Thus there is still some hope that the conjecture of Effros and Shen is valid for a rather big class of finitely generated dimension groups.

REFERENCES


Institut für Mathematik, Technische Universität, D-8000 München, Federal Republic of Germany

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use