PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 1, September 1981

ON THE ANALYTIC COHOMOLOGY OF A DOMAIN IN A STEIN MANIFOLD

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ABSTRACT. Suppose M is an open subset of a Stein manifold without isolated points and that V is a holomorphic vector-bundle on M admitting a holomorphic connection. Then $H^q(M, \mathcal{O}(V))$ is either zero or of infinite dimension.

In [1] Laufer showed that, for M an open subset of a Stein manifold without isolated points, $H^q(M, \Omega^p)$ is either zero or of infinite dimension as a complex vector-space. Here Ω^p denotes the sheaf of germs of holomorphic p-forms. In this note we extend this result to $H^q(M, \mathcal{O}(V))$ for any holomorphic vector-bundle V obtained as the restriction of a holomorphic vector-bundle on the ambient Stein manifold (see corollary below). The argument is an extension of that given by Laufer for $H^q(M, \mathcal{O})$. The crucial ingredient is a connection on V (see theorem below).

Since making this extension I have learned that these results were also obtained by John Sensat, a student of Professor Laufer. Unfortunately, he was killed in an accident approximately three years ago. I respectfully dedicate this paper to his memory.

LEMMA. *Suppose V is a holomorphic vector-bundle on a complex manifold M. Suppose V admits a holomorphic connection. Fix a nonnegative integer q and let J denote the ideal of holomorphic functions on M which induce (by multiplication on the sheaf level) the zero map of $H^q(M, \mathcal{O}(V))$ to itself. Then $f \in J \Rightarrow Xf \in J$ for any holomorphic vector-field X on M.*

PROOF. Let $\nabla\colon \mathcal{O}(V) \to \Omega^1(V)$ be a connection. By the Leibniz rule $(Xf)s = \nabla_X(fs) - f\nabla_X s$ for $s \in \mathcal{O}(V)$. By functoriality the same is true for $s \in H^q(M, \mathcal{O}(V))$ and the result follows.

THEOREM. *Suppose M is an open subset of a Stein manifold without isolated points. Suppose V is a holomorphic vector-bundle on M that admits a holomorphic connection. Then $H^q(M, \mathcal{O}(V))$ is either zero or of infinite dimension.*

PROOF. Denote by S the ambient Stein manifold and regard S as being embedded in $\mathbf{C}^N$. Let I denote the ideal of holomorphic functions on S which induce the zero map on $H^q(M, \mathcal{O}(V))$, i.e., using the notation of the lemma above, $f \in I \Leftrightarrow f|_M \in J$. Suppose $H^q(M, \mathcal{O}(V))$ is of finite dimension. It suffices to show

Received by the editors November 15, 1980.
1980 *Mathematics Subject Classification*. Primary 32C35; Secondary 32C10.

that $1 \in I$. The coordinate functions z_j on $\mathbf{C}^N$ induce linear endomorphisms of $H^q(M, \mathcal{O}(V))$. Let p_j be the minimal polynomial of z_j regarded as such an endomorphism. Then $p_j(z_j) \in I$ and the collection $p_1(z_1), \ldots, p_N(z_N)$ have only finitely many common zeros on S. However, for any $x \in S$ we can choose $f \in I$ with $f(x) \neq 0$, for first choose any nonzero $f \in I$ and then use the lemma above repeatedly to reduce the order of the zero. Here we are using Cartan's Theorem B to assert that we can specify the value at x of a holomorphic vector-field on S. Thus we can augment our collection $p_j(z_j)$ to a collection, say $f_1, \ldots, f_n \in I$, with no common zeros. Now Theorem B shows that we can find $g_1, \ldots, g_n \in \Gamma(S, \mathcal{O})$ with $\Sigma\, g_j f_j = 1$ and so $1 \in I$ as required.

COROLLARY. *Suppose M is an open subset of a Stein manifold S without isolated points. Suppose V is a holomorphic vector-bundle on S. Then $H^q(M, \mathcal{O}(V))$ is either zero or of infinite dimension.*

PROOF. V admits a holomorphic connection on S since the obstruction lies in $H^1(S, \Omega^1(\mathrm{End}\, V))$ and this vanishes by Cartan's Theorem B. Now restrict to M and apply the theorem.

REFERENCES

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