DERIVATIVES OF POLYNOMIALS WITH POSITIVE COEFFICIENTS

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Abstract. Let \( P_n(x) \) be an algebraic polynomial of degree \( n \) with positive coefficients. We set

\[
I_n = \frac{\|P_n'(x)\omega(x)\|_{L_2(0, \infty)}}{\|P_n(x)\omega(x)\|_{L_2(0, \infty)}}.
\]

In this work upper bounds of \( I_n \) are investigated. We restrict ourselves here with the case \( \omega(x) = x^{a/2}e^{-x/2} \). Results are shown to be best possible.

1. The problems dealt with in the present paper are connected with the classical inequalities of A. Markov [5], P. Erdos [1], G. G. Lorentz [3], [4], G. Szego [7] and P. Turan [8]. First, we describe them briefly.

**Theorem A (A. A. Markov).** If \( f(x) \) is a polynomial of degree \( n \), such that \( |f(x)| < 1 \) for \( a < x < b \), then

\[
\max_{a < x < b} |f'(x)| < \frac{2}{b - a} n^2.
\]

Further, this bound is sharp, as can be seen by the example

\[
f(x) = T_n(2(x - a)/(b - a) - 1)
\]

where \( T_n \) is Chebyshev’s polynomial.

For special polynomials, one can sometimes improve the estimate (1.1). Next, two theorems are of this kind.

**Theorem B (P. Erdos).** Let \( f(x) \) be a polynomial of degree \( n \) satisfying the inequality \( |f(x)| < 1 \) for \(-1 < x < 1\). Suppose \( f(x) \) has only real roots and no root inside \([-1, +1]\); then for \(-1 < x < 1\),

\[
|f'(x)| < \frac{1}{2} en.
\]

This is the best possible result.

Inequalities of the same type (as in Theorem B) hold for the wider class of polynomials with positive coefficients in \( 1 + x, 1 - x \); that is, for the polynomials

\[
f(x) = \sum_{k + l \leq n} a_{kl}(1 + x)^k(1 - x)^l, \quad a_{kl} > 0, \quad -1 < x < 1.
\]

These polynomials were introduced by G. G. Lorentz [3] and studied extensively by J. T. Scheick [6]. The Lorentz theorem can be stated as follows.

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Theorem C (G. G. Lorentz). There exists a constant $C > 0$ such that for each polynomial $f(x)$ of the form (1.3),

$$\frac{\|f'\|}{\|f\|} \leq Cn, \quad n = 1, 2, \ldots,$$

for the uniform norm on $[-1, +1]$.

In 1964 G. Szego [7] studied the order of magnitude of $\frac{\|f'\|}{\|f\|}$ for unrestricted polynomials $f$ of degree $\leq n$ for the norm

$$\|f\| = \sup_{x>0} |f(x)e^{-x}|$$
on $(0, \infty)$. More precisely, he proved the following.

Theorem D (G. Szego). Let $f(x)$ be a polynomial of fixed degree $n$ and not vanishing identically. Then

$$\frac{\|f'\|}{\|f\|} \leq Cn, \quad n = 2, 3, \ldots,$$

holds (for the weighted uniform norm as defined by (1.5) on $(0, \infty)$).

In 1968 G. G. Lorentz [4] considered the problem of G. Szego for the special polynomials with positive coefficients in $x$,

$$f(x) = \sum_{k=0}^{n} a_k x^k, \quad a_k > 0,$$

and the norm of a function $f$ on $(0, \infty)$ is given by

$$\|f\| = \sup_{x>0} |f(x)e^{-\omega(x)}|,$$

where $\omega$ increases on $(0, \infty)$. Then the Lorentz theorem can be stated as follows.

Theorem E (G. G. Lorentz). Let $\omega(x)$ satisfy the inequalities $\omega(x) - \omega(0) < A\omega'(x), x > 0$, and $\omega'(y) < A\omega'(x), y < x$, for some constant $A > 0$. Then for some constant $C > 0$, the inequality

$$\frac{\|f'\|}{\|f\|} \leq C \frac{\|f'_n\|}{\|f_n\|}, \quad f_n(x) = x^n,$$

is valid.

Other related interesting results are due to A. Zygmund [10], Hille, Szego and Tamerkin [2] and P. Turan [8], where similar problems are considered either in $L_2$ norm or even in $L_p$ norm.

We may note that Lorentz raised the problem of determining the best constant in (1.6) and for (1.9).

Now we state the main theorem of this paper. Let $S_n[a, b]$ be the set of all polynomials whose degree is $n$ and whose roots are all real and no root inside $(a, b)$. It is easy to see that if $p_n \in S_n(0, \infty)$ then it can be expressed in the form

$$p_n(x) = \sum_{k=0}^{n} a_k x^k, \quad a_k > 0 \text{ for } k = 0, 1, \ldots, n.$$
We now state

**Theorem 1.** Let $P_n(x)$ be an algebraic polynomial of degree $n$ with nonnegative coefficients. Then for $\alpha > (\sqrt{5} - 1)/2$

\[
\int_0^\infty (P_n'(x))^2 x^{\alpha} e^{-x} \, dx \leq \frac{n^2}{(2n + \alpha)(2n + \alpha - 1)} \int_0^\infty P_n^2(x) x^{\alpha} e^{-x} \, dx,
\]

equality holding for $P_n(x) = x^n$. For $0 < \alpha < \frac{1}{2}$ we have

\[
\int_0^\infty (P_n'(x))^2 x^{\alpha} e^{-x} \, dx < \frac{1}{(2 + \alpha)(1 + \alpha)} \int_0^\infty P_n^2(x) x^{\alpha} e^{-x} \, dx.
\]

Moreover (1.10) is also best possible in the sense that for $P_n(x) = x^n + \lambda x$ the expression on the left can be made arbitrarily close to the expression on the right-hand side by choosing $\lambda$ positive and sufficiently large.

**Corollary.** Let $P_n(x)$ be an algebraic polynomial of degree $n$ with nonnegative coefficients satisfying (1.8) with $a_n > 0$. Then for $\alpha > (\sqrt{5} - 1)/2$ we obtain

\[
\int_0^\infty P_n^2(x) x^{\alpha} e^{-x} \, dx \geq (a_n)^2 \Gamma(2n + \alpha + 1),
\]

equality holding for $P_n(x) = x^n$ ($a_n = 1$).

Proof of this Corollary is as follows. Repeated application of the inequality (1.9) gives at once

\[
\int_0^\infty (P_n^k(x))^2 x^{\alpha} e^{-x} \, dx < \frac{n^2(n - 1)^2 \cdots (n - k + 1)^2}{(2n + \alpha)(2n + \alpha - 1) \cdots (2n + \alpha - 2(k - 1))(2n + \alpha - 1 - 2(k - 1))}.
\]

Next, we note that

\[
\int_0^\infty (P_n^{(n)}(x))^2 x^{\alpha} e^{-x} \, dx = (a_n)^2 (n!)^2 \Gamma(\alpha + 1).
\]

Now, we substitute $k = n$ in (1.12) and use (1.13) which leads to (1.11).

2. **Proof of Theorem 1.** Let $P_n(x)$ be any algebraic polynomial of degree $n$ with positive coefficients. Then we may write

\[
P_n(x) = a_n x^n + P_{n-1}(x), \quad P_{n-1}(x) = \sum_{k=0}^{n-1} a_k x^k, \quad a_k > 0.
\]

Using (2.1) and

\[
\int_0^\infty x^k e^{-x} \, dx = \Gamma(k + 1)
\]
we obtain

\[ \int_0^\infty (P'_n(x))^2 e^{-x^a} \, dx = \int_0^\infty (P'_{n-1}(x))^2 e^{-x^a} \, dx + a_n^2 n^2 \Gamma(2n + \alpha - 1) + 2n a_n \int_0^\infty P'_{n-1}(x) x^{n+a-1} e^{-x} \, dx, \]

and

\[ \int_0^\infty (P_n(x))^2 e^{-x^a} \, dx = a_n^2 \Gamma(2n + \alpha + 1) + \int_0^\infty (P'_{n-1}(x))^2 e^{-x^a} \, dx + 2a_n \int_0^\infty P_{n-1}(x) e^{-x^{n+a}} \, dx. \]

Next we set

\[ \lambda_n = 2n \int_0^\infty P'_{n-1}(x) x^{n+a-1} e^{-x} \, dx - b_n \int_0^\infty P_{n-1}(x) x^{n+a} e^{-x} \, dx. \]

From (2.3)–(2.6) we may conclude that

\[ \int_0^\infty (P'(x))^2 e^{-x^a} \, dx - b_n \int_0^\infty (P_n(x))^2 e^{-x^a} \, dx = \lambda_n a_n + \int_0^\infty (P'_{n-1}(x))^2 e^{-x^a} \, dx - b_n \int_0^\infty (P_{n-1}(x))^2 e^{-x^a} \, dx. \]

First, we claim that for \( a > 0 \)

\[ \lambda_n < 0, \quad n = 1, 2, \ldots. \]

From (2.1), (2.2) and (2.6) we obtain

\[ \lambda_n = \frac{2n}{(2n + \alpha)(2n + \alpha - 1)} \sum_{k=0}^{n-1} a_k \mu_{kn} \Gamma(k + n + \alpha - 1), \quad a_k > 0, \]

where

\[ \mu_{kn} = (2n + \alpha)(2n + \alpha - 1)k - n(k + n + \alpha)(k + n + \alpha - 1) \]

\[ = (k - n)(n(n - k) + (2\alpha - 1)n + \alpha(\alpha - 1)), \quad 0 < k < n - 1. \]

Clearly for \( a > 0 \) we have

\[ \mu_{kn} < -a(2n + \alpha - 1) < 0, \quad k = 0, 1, \ldots, n - 1. \]

Thus (2.8) follows easily from (2.9) and (2.10). We also note that

\[ b_n > b_{n-1}, \quad a > \frac{\sqrt{5} - 1}{2}, \quad n = 2, 3, \ldots, \]

and

\[ b_n < b_{n-1}, \quad 0 < a < \frac{1}{2}, \quad n = 2, 3, \ldots. \]
Using these ideas we can complete the proof of Theorem 1. For \( \alpha > (\sqrt{5} - 1)/2 \) we use (2.7), (2.8) and (2.11) and we obtain for \( k = 2, 3, \ldots, n \),

\[
\int_0^\infty (P_k'(x))^2 e^{-x^\alpha} \, dx - b_k \int_0^\infty (P_k(x))^2 e^{-x^\alpha} \, dx
\]

Adding all these equations we obtain

\[
\int_0^\infty (P_0(x))^2 e^{-x^\alpha} \, dx - b_0 \int_0^\infty (P_0(x))^2 e^{-x^\alpha} \, dx < \int_0^\infty (P_{k-1}(x))^2 e^{-x^\alpha} \, dx - b_{k-1} \int_0^\infty (P_{k-1}(x))^2 e^{-x^\alpha} \, dx.
\]

But a simple computation shows that if \( P_1(x) = a_1 x + a_0, a_1 > 0, a_0 > 0 \) then

\[
\int_0^\infty (P_1'(x))^2 e^{-x^\alpha} \, dx < b_1 \int_0^\infty (P_1(x))^2 e^{-x^\alpha} \, dx,
\]

equality holds if \( P_1(x) = a_1 x, a_1 > 0 \). From (2.14) and (2.15) we obtain (1.2) as desired for \( \alpha > (\sqrt{5} - 1)/2 \). For the proof of (1.3) for \( 0 < \alpha < \frac{1}{2} \) we use (2.7), (2.8) and (2.12). From these results it follows that

\[
\int_0^\infty (P_k'(x))^2 e^{-x^\alpha} \, dx - b_k \int_0^\infty (P_k(x))^2 e^{-x^\alpha} \, dx < \int_0^\infty (P_{k-1}(x))^2 e^{-x^\alpha} \, dx - b_{k-1} \int_0^\infty (P_{k-1}(x))^2 e^{-x^\alpha} \, dx
\]

Also from (2.4) it follows that

\[
\int_0^\infty P_{k-1}(x)^2 e^{-x^\alpha} \, dx < \int_0^\infty (P_k(x))^2 e^{-x^\alpha} \, dx.
\]

From (2.16) and (2.17) we obtain for \( k = 2, 3, \ldots, n - 1, n \),

\[
\int_0^\infty (P_k'(x))^2 e^{-x^\alpha} \, dx - b_k \int_0^\infty (P_k(x))^2 e^{-x^\alpha} \, dx < \int_0^\infty (P_{k-1}(x))^2 e^{-x^\alpha} \, dx - b_{k-1} \int_0^\infty (P_{k-1}(x))^2 e^{-x^\alpha} \, dx
\]

We add all these equations for \( k = 2, 3, \ldots, n - 1, n \) and we obtain

\[
\int_0^\infty (P_n'(x))^2 e^{-x^\alpha} \, dx - b_n \int_0^\infty (P_n(x))^2 e^{-x^\alpha} \, dx < \int_0^\infty (P_{n-1}(x))^2 e^{-x^\alpha} \, dx - b_{n-1} \int_0^\infty (P_{n-1}(x))^2 e^{-x^\alpha} \, dx
\]
But clearly (2.18) is equivalent to

\[(2.20) \quad \int_0^\infty (P_n(x))^2 e^{-x^\alpha} \, dx \leq b_n \int_0^\infty (P_n(x))^2 e^{-x^\alpha} \, dx.\]

From (2.19) and (2.5) follows (1.3).

In order to show that (1.3) is best possible we consider \(f_0(x) = x^n + \lambda x, \lambda > 0, 0 < \alpha \leq \frac{1}{2}\). A simple computation shows that

\[
\frac{\int_0^\infty (f'_0(x))^2 e^{-x^\alpha} \, dx}{\int_0^\infty (f_0(x))^2 e^{-x^\alpha} \, dx} = \frac{1}{(\alpha + 2)(\alpha + 1)} - C_n\lambda,
\]

where

\[C_n\lambda = \frac{2\lambda \Gamma(n + \alpha)\left[\alpha(\alpha + 1) + n^2 - n(\alpha^2 + \alpha - 1)\right] + d_n}{(\alpha + 2)(\alpha + 1)\left[\lambda^2 \Gamma(\alpha + 3) + 2\lambda \Gamma(n + \alpha + 2) + \Gamma(2n + \alpha + 1)\right]},\]

and

\[d_n = \Gamma(2n + \alpha - 1)\left[n^2(2 - 3\alpha - \alpha^2) + 2n(2\alpha - 1) + \alpha(\alpha - 1)\right].\]

Clearly \(C_n\lambda \to 0\) as \(\lambda \to \infty\) from which our assertion follows.

REFERENCES


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