RIEMANN $R_1$-SUMMABILITY OF INDEPENDENT, IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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Abstract. Let $X, X_1, X_2, \ldots$ be i.i.d. random variables. It is shown that $E|X|\log^+\log^+|X| < \infty$ is a sufficient condition for Riemann $R_1$-summability of \{X\}_{n=1}^\infty$ to $EX$. Counterexamples are provided which indicate that the strongest possible necessary condition of moment type is $E|X| < \infty$. However under weak regularity conditions on the tails of the distribution of $X$ the sufficient condition is also shown to be necessary.

1. Introduction. Many regular forms of summability have been studied in great detail for independent, identically distributed random variables $X, X_1, X_2, \ldots$; the classic example being the strong law of large numbers which asserts that $\sum|X_i| < \infty$ iff \{X\}_{n=1}^\infty is Cesàro ($C, 1$) summable to $EX$. Lai [5] has shown that the existence of the first absolute moment is also necessary and sufficient for Cesàro ($C, \alpha$) summability for all $\alpha > 1$ as well as for Abel summability, whereas Chow [1] in studying Euler ($E, q$), $q > 0$, summability and Borel summability found that the existence of a second moment was equivalent to summability. However, results for Riemann $R_1$-summability are not so tidy. Recall that a sequence of real numbers \{A_n\} is $R_1$-summable to $A$ if

$$\lim_{t \to 0^+} \frac{2}{\pi} \sum_{n=1}^{\infty} n^{-1} A_n \sin nt = A. \quad (1)$$

It is well known [3] that this is not a regular summability method. More importantly from our point of view, it is a discrete to continuous method in that the limit in (1) is taken through a continuum of values. In a recent paper Cuzick and Lai [2] have shown that a sequence of i.i.d. random variables $X, X_1, X_2, \ldots$ is $R_1$-summable to $EX$ if $E|X|\log^+|X| < \infty$ and also that $E|X| < \infty$ is a necessary condition for summability. The approach there, as also followed here, is to treat (1) as a random Fourier series and, when $EX = 0$, to establish uniform convergence. The purpose of the present note is threefold:

(i) to show that

$$E|X|\log^+\log^+|X| < \infty \quad (2)$$

is a sufficient condition for $R_1$-summability,

(ii) to give examples which demonstrate that in general the condition $E|X| < \infty$ is the strongest possible necessary condition of moment type, i.e., given a monotone
function \( g(x) \) such that \( \lim_{x \to \infty} g(x)/x = +\infty \) we can find an \( R_1 \)-summable random variable \( X \) with \( E\)\(|X|\) = +\( \infty \), and

(iii) to show that under a weak regularity condition on the smoothness of the tails of the distribution function of \( X \) the condition (2) is in fact necessary and sufficient.

2. Sufficient conditions. As the constant sequence is \( R_1 \)-summable we may take \( EX = 0 \). Then \( R_1 \)-summability follows from the continuity of

\[
S(t) = \sum_{n=1}^{\infty} n^{-1}X_n \sin nt
\]

which in turn follows from the uniform convergence of the series in (3).

**Theorem 1.** Assume \( EX = 0 \) and \( E|X|\log^+\log^+ < \infty \), where \( \log^+x = \max(0, \log x) \). Then the series in (3) is uniformly convergent and \( \{X_n\} \) is \( R_1 \)-summable.

**Proof.** The proof is a refinement of the techniques used in [2]. Let \( X \) have distribution function \( F \) and for \( \beta > 1 \) set

\[
X'_n = X_n 1_{|X_n| < n/\log^\beta n}, \quad X''_n = X_n 1_{n/\log^\beta n < |X_n| < n},
\]

\[
X'''_n = X_n 1_{|X_n| > n}
\]

and write

\[
S(t) = \sum_{n=1}^{\infty} (X'_n - EX'_n)n^{-1} \sin nt + \sum_{n=1}^{\infty} EX'_n n^{-1} \sin nt
\]

\[
+ \sum_{n=1}^{\infty} X''_n n^{-1} \sin nt + \sum_{n=1}^{\infty} X'''_n n^{-1} \sin nt
\]

\[
= S_1(t) + S_2(t) + S_3(t) + S_4(t), \quad \text{say.}
\]

As \( E|X| < \infty \) it follows from the Borel-Cantelli lemma that \( S_4 \) has only a finite number of nonzero terms and hence is uniformly convergent. Uniform convergence of \( S_3 \) follows from the fact that

\[
E \sum_{n=3}^{\infty} n^{-1}|X''_n| = \sum_{n=3}^{\infty} n^{-1} \int_{n/\log^\beta n < |x| < n} |x| dF(x)
\]

\[
< \text{Const.} \int_{-\infty}^{\infty} \left( \sum_{|x| < n < |x|\log^\beta |x|} n^{-1} \right) |x| dF(x)
\]

\[
< \text{Const.} \int_{-\infty}^{\infty} |x|\log^+\log^+ |x| dF(x) < \infty.
\]

The uniform convergence of \( S_1(t) \) follows from a result of Marcus [7] (see also [2, remarks after Theorem 3]) by checking that

\[
\sum_{n=3}^{\infty} \left\{ \left( \sum_{k=n}^{\infty} E(X'_k)^2 k^{-2} \right)^{1/2} / n \log^{1/2} n \right\} < \infty
\]
which follows from the estimate
\[
\sum_{k=n}^{\infty} E(X_n^2) k^2 = \sum_{k=n}^{\infty} k^{-2} \int_{|x| < k/\log^\beta n} x^2 dF(x)
\]
\[
= \int_{|x| < n/\log^\beta n} \left( \sum_{k=n}^{\infty} k^{-2} \right) x^2 dF(x)
\]
\[
+ \int_{|x| > n/\log^\beta n} \left( \sum_{k=n}^{\infty} k^{-2} \right) x^2 dF(x)
\]
\[
= O(\log n)^{-\beta} \quad \text{since } \beta > 1.
\]
As $EX = 0$, we may replace $EX_n'$ by $EX|X| > n/\log^\beta n$ in $S_2$. Interchanging summation and expectation as before we find the $N$th partial sum of $S_2(t)$ equals
\[
\int_{-\infty}^{\infty} \left( \sum_{n/\log^\beta n < x} n^{-1} \sin nt \right) x dF(x). \quad (4)
\]
If we compute
\[
C(i, t) = \sup_{j} \left| \sum_{i < n < j} n^{-1} \sin nt \right|
\]
and
\[
C = \sup_{i, t} C(i, t),
\]
it is easy to check that $C < \infty$ and $C(i, t) \downarrow 0$ as $i \uparrow \infty$ uniformly for $t$ on compact sets not containing $(2\pi n)^{\infty}_{n=\infty}$. From this it follows that $S_2(t)$ is uniformly convergent on such sets and we need only concern ourselves with uniform convergence in a neighborhood of the origin, i.e. given $\epsilon > 0$, it is enough to find a $\delta > 0$ such that when $|t| < \delta$, (4) is less than $\epsilon$ for all $N$. Choose $D$ so that
\[
\int_{|x| > D} |x| dF(x) < \epsilon / 2C.
\]
Then the integral (4) on the set $|x| > D$ is less than $\epsilon / 2$ and it suffices to show that for all $N$
\[
\int_{|x| < D} \left( \sum_{n/\log^\beta n < x} n^{-1} \sin nt \right) x dF(x) < \epsilon / 2
\]
for all $t$ sufficiently small. This follows immediately from the convergence to zero of the integrand as $t \to 0$ uniformly for $|x| < D$, as the series contains only a finite number of terms for $|x|$ bounded.

3. Necessary conditions. Cuzick and Lai [2] noted that $E|X| < \infty$ was necessary for $R_1$-convergence. The following example indicates that no stronger necessary condition of moment type can be found. Let $g(x)$ be any function such that $g(x)/x \uparrow \infty$ as $x \to \infty$. Choose any $0 < \epsilon < 1$ and let $n_k$ be the first $n > 2n_k-1$ and of the form $2j/n$ for some $j > 1$ such that $g(n_k)/n_k > k^{2+\epsilon}$. Let $X$ have mass $M/(n_k k^{2+\epsilon})$ at $\pm n_k$ with $M$ chosen so that the total mass is unity. Then $E|X| < \infty$ but $Eg(|X|) = +\infty$. As $E|X_n| < \infty$, by the Borel-Cantelli lemma we may replace...
Now for \( n_k < 2^n < n_{k+1} \) compute

\[
s_n = \left( \sum_{j=2^{n+1}}^{2n+1} E(X'_j/j)^2 \right)^{1/2} = O\left( \frac{n_k}{2^n k^{2+\epsilon}} \right)^{1/2}.
\]

Then we may choose \( s^*_n > s_n \), with \( s^*_n \) nonincreasing so that

\[
\sum_{n=1}^{\infty} s^*_n = \sum_{n=1}^{\infty} \left( \frac{k}{n_k} \sum_{k=2^n < n_{k+1}} s^*_n \right) = O\left( \sum_{k=1}^{\infty} k^{-(1+\epsilon/2)} \right) < \infty.
\]

It follows from a result of Kahane [4, p. 65, Remark 2] that

\[
\sum_{n=1}^{\infty} X'_n/n^{-1} \sin nt
\]

is uniformly convergent and thus \( \{X'_n\} \) is \( R_1 \)-summable.

This result relies heavily on the lacunary behavior of the distribution of \( X \). If some mild conditions are placed on the regularity of the tails of the distribution of \( X \), it is possible to show that (2) is also necessary for \( R_1 \)-summability.

**Theorem 2.** Assume that for all \( n \) sufficiently large

\[
\int_{2^n < |x| < 2^{n+1}} |x| \ dF(x) \text{ is nonincreasing as } n \uparrow \infty. \quad (5)
\]

Then \( \{X'_n\} \) is Riemann \( R_1 \)-summable \( \iff \ E|X| \log^+ \log^+ |X| < \infty. \)

**Proof.** We need only establish that under (5), condition (2) is necessary for summability. First assume \( X \) is symmetric. We shall need the results of §3 in [2]. Set \( W_n = XI_{0 < x < n} \) and let \( G_n \) be the unique solution of the equation

\[
nE\left( \min\{ |W_n/G_n|, (W_n/G_n)^2 \} \right) = 1. \quad (6)
\]

If \( \{X'_n\} \) is summable it follows from Theorem 5 of [2] that

\[
\sum_{n=1}^{\infty} \gamma_n < \infty \quad (7)
\]

where \( \gamma_n = G_{2^n}/2^n. We aim to show that

\[
\gamma_n = o\left( \frac{1}{n} \right) \quad (8)
\]

which follows from (7) if we can establish that \( \gamma_n \) is “almost monotone” in the sense that there exists an \( n_0 \) and \( K_1 > 0 \) such that

\[
\gamma_n > K_1 \gamma_m \quad \text{for all } m > n > n_0. \quad (9)
\]

To prove this we need to verify the existence of an \( n_0 \) and \( K_2 > 0 \) such that for all \( n > j > n_0 \)

\[
2^{-n} \int_0^{2^n} x^2 \ dF(x) > K_2 2^{-(n+k)} \int_0^{2^n+k} x^2 \ dF(x), \quad \text{all } k > 0 \quad (10)
\]

and

\[
\int_{2^j}^{2^{j+k}} x \ dF(x) > K_2 \int_{2^j+k}^{2^{j+k}} x \ dF(x), \quad \text{all } k > 0. \quad (11)
\]
Equation (11) follows immediately from (5) and (10) follows from (5) upon noting that
\[
\int_{2^{2n+1}}^{2^{2n}x} x^2 dF(x) = \sum_{m=n_0}^{n} \int_{2^{2m-1}}^{2^{2m-1}x} x^2 dF(x) > \sum_{m=n_0}^{n} 2^{m-1} \int_{2^{2m-1}x}^{2^{2m-1}x} x dF(x) > \sum_{m=n_0}^{n} 2^{m-1} \int_{2^{2m-1}x}^{2^{2m-1}x} x dF(x)
\]
\[
> \sum_{m=n_0}^{n} 2^{k-2} \int_{2^{2m+k-1}x}^{2^{2m+k-1}x} x^2 dF(x) > \frac{1}{4} 2^{-k} \int_{2^{2n+1}}^{2^{2n+1}x} x^2 dF(x).
\]

Now for \(0 < G < 2^n\), define
\[
g_n(G) = \int_0^G \frac{x^2}{G} dF(x) + \int_G^{2^n} x dF(x).
\]
(12)

Then \(g_n\) is nonincreasing in \(G\). Choose \(\beta_n\) of the form \(2^i\) for some integer \(i\) such that \(\frac{1}{2} \beta_n < G_{2^i} < \beta_n\). Then using (6)
\[
\gamma_n = g_n(G_{2^i}) > g_n(\beta_n)
\]
(13)

which from (10) and (11) is greater than or equal to \(K_2 g_{n+k}(2^k \beta_n)\) for all \(k > 0\). If \(2^k \beta_n < G_{2^i+k}\) then
\[
K_2 g_{n+k}(2^k \beta_n) > K_2 g_{n+k}(G_{2^i+k}) = K_2 \gamma_{n+k}
\]
so that (9) holds. Of course (9) also holds when \(2^k \beta_n > G_{2^i+k}\) since then \(G_{2^i} > \frac{1}{2} \beta_n > 2^{-(k+1)} G_{2^i+k}\) so that \(\gamma_n = G_{2^i}/2^n > \frac{1}{2} \gamma_{n+k}\).

Now define \(G(x) = G_{2^n}\) for \(x = n\) and by linear interpolation for nonintegral \(x\). It is easily checked that \(G\) is nondecreasing. From (8) it follows that there exists \(K_3 > 0\), such that for sufficiently large \(x\)
\[
G^{-1}(x) > \log x + K_3 \log \log x.
\]
(14)

It follows from (7), (13) and (12) that
\[
\sum_{n=1}^{\infty} \int_{G(n)}^{2^n} x dF(x) < \infty
\]
and a Fubini argument shows
\[
\int_0^{\infty} \left[ G^{-1}(x) - \log x \right] x dF(x) < \infty
\]
from whence (2) follows with the help of (14). This establishes the theorem when \(X\) is symmetric. In general let \(m\) be a median of \(X\) and for \(2^k < n < 2^{k+1}\) define
\[
Y_n = (X_n - m) I_{|X_n - m| < 2^{k-1}}
\]
and let \(Z_n\) be the symmetrized version of \(Y_n\), i.e. \(Z_n = Y_n - Y'_n\) where \(Y_n, Y'_n\) are i.i.d. Then \(\{X_n\}\text{-summable} \Rightarrow \{Z_n\}\text{-summable}
\[
\Rightarrow \sum_{k=1}^{\infty} \gamma_k Z^k < \infty \text{ as at (7)}
\]
(15)
where \(\gamma_k = G_{2^i}/2^k\) and \(G_{2^i}\) is the unique solution of (6) with \(W_n\) replaced by \(Z_{2^k}\).
Define $\gamma_k^Y$ and $G_k^X$ similarly. Then integration by parts and the weak symmetrization inequality \[6, p. 245\] gives

\[
\gamma_k^Z = E(\min\{Z_k^2/G_{k2}, |Z_k^2|\}) = \int_0^\infty \min\{x^2/G_{k2}, x\} \, dF_{|Z_k^2|}(x)
\]

\[
> \frac{1}{2} \int_0^\infty (1 - F_{|Z_k^2|}(x))\min\{2x/G_{k2}, 2\} \, dx
\]

\[
> \frac{1}{4} \int_0^\infty (1 - F_{|Y_k^2|}(\frac{x}{2}))\min\{2x/G_{k2}, 2\} \, dx
\]

\[
> \frac{1}{16} \int_0^\infty (1 - F_{|Y_k^2|}(x))\min\{2x/G_{k2}, 2\} \, dx
\]

\[
> \frac{1}{16} E(\min\{Y_k^2/G_{k2}, |Y_k^2|\}).
\]

From this it can be shown as in the argument following (13) that

\[
\gamma_k^Z > \frac{1}{8} \gamma_k^Y. \tag{16}
\]

It is easily verified from (6) that for $k$ large there exists $K_3 > 0$ such that

\[
G_k^X > K_3 2^{k/2}. \tag{17}
\]

Now for $G_k^X > 2m$ we have

\[
\gamma_k^Y = E(\min\{Y_k^2/G_{k2}, |Y_n^2|\})
\]

\[
= E(\min\{(X - m)^2 I_{|X-m| < 2^{k-1} / G_{k2}}, |X - m| I_{|X-m| < 2^{k-1}}\})
\]

\[
> \frac{1}{2} E(\min\{X^2 I_{|X| < 2^{k-1} / G_{k2}}, |X| I_{|X| < 2^{k-1}}\} - m^2 / G_{k2})
\]

\[
> \frac{1}{2} (\gamma_k^X - m^2 / G_{k2}).
\]

Combining (15), (16) and (17) gives

\[
\sum_{k=1}^\infty \gamma_k^X < \infty. \tag{18}
\]

If we let $|V| = |X|$ and let $V$ be symmetric, i.e. $V = \pm X$, then $\gamma_k^V = \gamma_k^X$ and, as $V$ satisfies (5) and is symmetric, we may use the proof previously given to see that (18) implies $\{V_n\}$ is summable

\[
=> E|X|\log^+\log^+|X| = E|V|\log^+\log^+|V| < \infty
\]

which completes the proof in general.

**References**


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