PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 1, September 1981

THE BOCKSTEIN AND THE ADAMS SPECTRAL SEQUENCES

J. P. MAY AND R. J. MILGRAM

ABSTRACT. We show that, above the appropriate "vanishing line", the Adams spectral sequence of a connective spectrum can be read off from its Bockstein spectral sequence.

In this short note, we prove a basic folklore theorem which relates the mod p homology Bockstein spectral sequence of X to the Adams spectral sequence $\{E_rX\}$ converging from $E_2X = \mathrm{Ext}_A(H^*X, Z_p)$ to π_*X where X is a bounded below spectrum with integral homology of finite type. As usual, we grade $\{E_rX\}$ so that

$$E_2^{s,t}X = \mathrm{Ext}_A^{s,t}(H^*X, Z_p),$$

with $d_r\colon E_r^{s,t}X \to E_r^{s+r,t+r-1}X$, the total degree being $t - s$. We have a natural homomorphism $E_2^{0,*}X \to H_*X$ which factors the mod p Hurewicz homomorphism, and we shall sometimes identify elements of H_*X with their inverse images in $E_2^{0,*}X$. We have a pairing of spectral sequences $E_rS \otimes E_rX \to E_rX$, where S is the sphere spectrum. Finally, we have an infinite cycle $a_0 \in E_2^{1,1}S$ such that if $x \in E_\infty^{s,t}X$ and if $y \in F^s\pi_{t-s}X$ projects to x, then $py \in F^{s+1}\pi_{t-s}X$ and py projects to a_0x.

Our main theorem will be a consequence of the following vanishing theorem, which is due to Adams [1] when $p = 2$ and to Liulevicius [4] when $p > 2$. Let $A_0 = E\{\beta\} \subset A$ and recall that an A_0-module M is free if and only if $H(M;\beta) = 0$.

THEOREM 1. *Let M be an $(m + 1)$-connected A_0-free A-module. Then* $\mathrm{Ext}_A^{s,t}(M, Z_p) = 0$ *for* $s > 1$ *and* $t - s < m + f(s)$, *where* $f(s) = 2(p - 1)s$ *if* $p > 2$ *and, if* $p = 2$, $f(4k) = 8k + 1$, $f(4k + 1) = 8k + 2$, $f(4k + 2) = 8k + 3$, *and* $f(4k + 3) = 8k + 5$.

DEFINITION 2. Let M be an A-module. We say that $x \in \mathrm{Ext}_A(M, Z_p)$ generates a spike if x is not of the form a_0x' and if $a_0^ix \neq 0$ for all i. The set of spikes in $\mathrm{Ext}_A(M, Z_p)$ has its evident meaning. The same language will be applied to each E_rX.

Let $K(R, n)$ denote the nth Eilenberg-Mac Lane spectrum of R and abbreviate $HR = K(R, 0)$. Let y denote the canonical generator of $H_0(HZ_{p^r})$ and let β_r denote the rth mod p Bockstein (in homology or cohomology according to context); let $y = \beta_r z$.

Received by the editors September 23, 1980.
1980 *Mathematics Subject Classification.* Primary 55T15; Secondary 55P42.

LEMMA 3. (i) *$E_2HZ_p = E_\infty HZ_p$ is Z_p in bidegree (0, 0).*

(ii) *For $r \geqslant 2$, $E_2HZ_{p^r}$ is the sum of a spike generated by $y \in E_2^{0,0}HZ_{p^r}$ and a spike generated by $z \in E_2^{0,1}HZ_{p^r}$; moreover, $d_r(a_0^iz) = a_0^{i+r}y$ and $E_\infty HZ_{p^r}$ has basis $\{a_0^iy \mid 0 \leqslant i < r\}$.*

(iii) *$E_2HZ = E_\infty HZ$ is a spike generated by $y \in E_2^{0,0}HZ$.*

PROOF. $H^*(HZ_p) = A \cdot \iota$, $H^*(HZ_{p^r}) = (A/A\beta) \cdot \iota \oplus (A/A\beta) \cdot \beta_r\iota$ for $r > 2$, and $H^*(HZ) = (A/A\beta) \cdot \iota$, where ι denotes the fundamental class. The calculation of the specified E_2 terms is immediate by change of rings [2, VI.4.13], and the differentials in (ii) follow (up to a nonzero constant) by convergence. That the constant is 1 can be checked by a comparison of the constructions of the Adams and the Bockstein spectral sequences.

We also record the following triviality.

LEMMA 4. *$E_r(X \vee Y) = E_rX \oplus E_rY$, with $d_r = d_r \oplus d_r$.*

Here, now, is the main result. Its proof derives from a discussion of the edge theorem one of us had with Mark Mahowald many years ago.

THEOREM 5. *Let X be an $(m-1)$-connected spectrum with integral homology of finite type. Let C_r, $r \geqslant 1$, be a basis for the rth term E^rX of the* mod p *homology Bockstein spectral sequence of X and assume the C_r chosen so that*

$$C_r = D_r \cup \beta_rD_r \cup C_{r+1},$$

where D_r, β_rD_r, and C_{r+1} are disjoint linearly independent subsets of E^rX such that $\beta_rD_r = \{\beta_rd \mid d \in D_r\}$ and C_{r+1} is a set of cycles under β_r which projects to the chosen basis for $E^{r+1}X$.

(i) *The set of spikes in E_rX, $2 \leqslant r$ and $r = \infty$, is in one-to-one correspondence with C_r; if $c \in C_r$ has degree q and $\gamma \in E_r^{s,t}X$ generates the corresponding spike, then*

$$f(s) + m \leqslant q = t - s.$$

(ii) *If $d \in D_r$ and if $\delta \in E_r^{s,t}X$ and $\varepsilon \in E_r^{u,v}X$, $v - u = t - s - 1$, generate the spikes corresponding to d and to β_rd, then*

$$d_r(a_0^i\delta) = a_0^{i+r+s-u}\varepsilon$$

provided that $m + f(i+s) \geqslant t - s$.

PROOF. Modulo torsion prime to p, $H_*(X; Z)$ is the direct sum of cyclic groups of order p^r whose generators reduce mod p to the elements of β_rD_r and of infinite cyclic groups whose generators reduce mod p to the elements of C_∞. By exploiting the universal coefficients theorem and the representability of integral and mod p^r cohomology, we can use this decomposition to construct maps

$$\phi_i\colon X \to K(H_i(X; Z), i)$$

which induce isomorphisms (modulo torsion prime to p) on integral homology in degree i. The map

$$\phi = \Sigma\,\phi_i\colon X \to \bigvee_i K(H_i(X; Z), i) \equiv Y$$

induces a monomorphism on mod p^r homology for all r. In particular, we have a short exact sequence

$$(*) \qquad 0 \to H_*(X; Z_p) \to H_*(Y; Z_p) \to M_* \to 0,$$

and closer inspection of the construction of the ϕ_i shows that

$$M_* \cong \sum_{q > i+2} H_q(K(H_i(X; Z), i); Z_p).$$

Since $H(A/A\beta; \beta) = 0$, we find (from the proof of Lemma 3) that the dual M of M_* is A_0-free and $(m + 1)$-connected. The exact sequence $(*)$ gives rise to a long exact sequence

$$\cdots \to \mathrm{Ext}_A^{s-1,t}(M, Z_p) \to E_2^{s,t}X \to E_2^{s,t}Y \to \mathrm{Ext}_A^{s,t}(M, Z_p) \to \cdots.$$

By Theorem 1, $E_2^{s,t}X \to E_2^{s,t}Y$ is an epimorphism if $s \geqslant 1$ and $t - s < m + f(s)$ and is an isomorphism if $s \geqslant 2$ and $t - s < m + f(s - 1)$. The conclusions follow directly from Lemmas 3 and 4, by naturality.

REMARK 6. Spikes of E_2X can be generated by elements lying in lower filtration degree than the range of isomorphism. Such generators can have nontrivial differentials earlier than predicted by the theorem (hitting classes annihilated by appropriate powers of a_0); in particular, such differentials can occur on the bottoms of spikes the top parts of which survive to $E_\infty X$. When X is a ring spectrum, such anomalous behavior is sometimes prevented by the relationship between the algebra structure of H_*X and its Bockstein spectral sequence.

We shall apply Theorem 5 to the study of the Adams spectral sequence converging to π_*MS Top in [3]. As will be illustrated there, the result can be a powerful tool for the computation of differentials in the Adams spectral sequence.

BIBLIOGRAPHY

1. J. F. Adams, *Stable homotopy theory*, Lecture Notes in Math., vol 3, Springer-Verlag, Berlin and New York, 1966.

2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956.

3. H. Ligaard, B. Mann, J. P. May and R. J. Milgram, *The odd primary torsion of the topological cobordism ring* (to appear).

4. A. Liulevicius, *Zeroes of the cohomology of the Steenrod algebra*, Proc. Amer. Math. Soc. **14** (1963), 972–976.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305