STRATIFIABLE SPACES AS SUBSPACES AND CONTINUOUS IMAGES OF $M_1$-SPACES

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Abstract. It is shown that every stratifiable space is the image of an $M_1$-space under a perfect retraction.

In 1961, J. Ceder introduced three classes of generalized metric spaces that he called $M_i$-spaces, $i = 1, 2, 3$. Ceder showed that every $M_1$-space is an $M_2$-space and every $M_2$-space is an $M_3$-space, and he asked whether the converse results also hold [3]. C. Borges renamed $M_3$-spaces “stratifiable spaces” and proved several important results concerning these spaces in [1]. One half of Ceder’s question was answered by G. Gruenhage and the second author of this paper, who showed, independently, that every stratifiable space is an $M_2$-space ([4] and [6]); the other half of the question has so far remained unanswered (for some partial answers, see [8] and [5]).

While the precise relationship between stratifiable spaces and $M_1$-spaces is yet to be determined, the following result sheds some light on that relationship.

Theorem. Every stratifiable space is the image of an $M_1$-space under a perfect retraction.

Proof. Let $X$ be a stratifiable space. We assume that $X$ is a $T_1$-space; it is easy to modify the following proof so that it works also for non-$T_1$-spaces (note that stratifiable spaces are regular and hence “essentially $T_1$”).

Denote by $S$ the subspace $\{0\} \cup \{1/n | n \in \mathbb{N}\}$ of the real line, and denote by $Z$ the space whose ground-set is the product-set $X \times S$ and whose topology is obtained by enlarging the product topology so as to make every point of the set $X \times \{1/n | n \in \mathbb{N}\}$ isolated.

It is not difficult to see that $Z$ is an $M_1$-space and that the projection $X \times S \rightarrow X$ is a retraction from $Z$ onto $X$; however, to obtain a perfect retraction, we must construct a suitable subspace of $Z$. Since $X$ is an $M_2$-space, there exists a family $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ of closed subsets of $X$ such that every point of $X$ has a neighborhood base consisting of members of $\mathcal{F}$ and, for every $n \in \mathbb{N}$, the family $\mathcal{F}_n$ is closure-preserving. We may assume $X \in \mathcal{F}_1$. It follows from Theorems 3.14 and 4.8 of [6] that, for every $n \in \mathbb{N}$, there exists a family $\mathcal{K}_n = \bigcup_{k \in \mathbb{N}} \mathcal{K}_{n,k}$ of subsets of $X$ such that, for each $k \in \mathbb{N}$, the family $\mathcal{K}_{n,k}$ is discrete, and, for each $x \in X$, there

Received by the editors April 25, 1980.

1980 Mathematics Subject Classification. Primary 54E20, 54C15, 54C25; Secondary 54C10, 54B55.

*While working on this paper, the second author was visiting the University of Pittsburgh as an Andrew Mellon Postdoctoral Fellow.

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0002-9939/81/0000-0433/$01.75
exists $K \in \mathcal{K}_n$ such that $x \in K \subset \bigcap \{F \in \bigcup_{m=1}^n \mathcal{F}_m | x \in F\}$. For each $K \in \bigcup_{n \in N} \mathcal{K}_n$, let $x_K$ be an element of $K$. For every $n \in N$, let

$$\mathcal{K}_n = \bigcup \{\mathcal{K}_{m,k} | m < n \text{ and } k < n\},$$

and let $A_n = \{x_K | K \in \mathcal{K}_n\}$; note that $\mathcal{K}_n$ is locally finite and, consequently, $A_n$ is discrete and closed. Let $Y$ be the subspace $(X \times \{0\}) \cup \bigcup_{n \in N} (A_n \times \{1/n\})$ of $Z$.

We show that $Y$ is an $M_1$-space. For all $F \in \mathcal{F}$ and $k \in N$, let

$$F_k = \left(\bigcap_{m=1}^k F \times \{0\}\right) \cup \left[\bigcap_{n=1}^m (F \times \{1/n| n > m\})\right].$$

We show that, for all $x \in X$, $F \in \mathcal{F}$ and $k \in N$, if $x \in F$, then $(x, 0) \in \text{Cl}_Y F_k$. Let $F \in \mathcal{F}$ and $k \in N$, and let $x \in F$. To show that $(x, 0) \notin \text{Cl}_Y F_k$, assume on the contrary that $(x, 0) \notin \text{Cl}_Y F_k$. Then there exists a neighborhood $G$ of $x$ in $X$ and $m \in N$ such that $(G \times \{1/n| n > m\}) \cap F_k = \varnothing$. Let $E \in \mathcal{F}$ be such that $x \in E \subset G$, and let $h \in N$ be such that $(E, F) \subset \bigcup_{n=1}^h \mathcal{F}_n$. There exists $K \in \mathcal{K}_n$ such that $x \in K \subset E \cap F$. Let $i \in N$ be such that $K \in \mathcal{K}_{h,i}$, and let $j = h + i + k + m$. Then $(x_K, 1/j) \in F_k$ and $(x_K, 1/j) \in \bigcap_{1/n| n > m} (F \times \{1/n| n > m\})$; this, however, is a contradiction since $(G \times \{1/n| n > m\}) \cap F_k = \varnothing$. It follows that $(x, 0) \notin \text{Cl}_Y F_k$.

For every $n \in N$, let $\mathcal{B}_n = \{F_k | F \in \mathcal{F}_n \text{ and } k \in N\}$. We show that, for each $n \in N$, the family $\mathcal{B}_n$ is closure-preserving in $Y$. Let $n \in N$ and $B \in \mathcal{B}_n$, and let $y \in \text{Cl}_Y B$. If $y \in X \times \{1/n| m \in N\}$, then $y$ is isolated and, consequently, $y \in \bigcup_{\mathcal{B}_n}$. Assume that $y = (x, 0)$ for some $x \in X$. Let

$$G = X \cup \{F \in \mathcal{F}_n | x \notin F\}.$$ 

Since $\mathcal{F}_n$ is closure-preserving in $X$, the set $G$ is a neighborhood of $x$ in $X$, and hence the set $G' = Y \cap (G \times S)$ is a neighborhood of $(x, 0)$ in $Y$. Since $(x, 0) \notin \text{Cl}_Y \bigcup_{\mathcal{B}_n}$, there exists $B \in \mathcal{B}_n$ such that $B \cap G' \neq \varnothing$. Let $F \in \mathcal{F}_n$ and $k \in N$ be such that $B = F_k$. We have $F_k \cap G' \neq \varnothing$ and consequently, $F \cap G \neq \varnothing$, that is $x \in F$. From the preceding part of the proof it follows that $(x, 0) \notin \text{Cl}_Y F_k$; in other words, that $y \in \text{Cl}_Y B$. We have shown that the families $\mathcal{B}_n$ are closure-preserving in $Y$. Clearly, the family $\mathcal{B}_0 = \{\{y\} | y \in Y(X \times \{1/n| n \in N\})\}$ is $\sigma$-closure-preserving in $Y$. It is easily seen that the family $\bigcup_{n=0}^\infty \mathcal{B}_n$ is a base for the topology of $Y$. Since $X$ is a regular space, so is $Y$. Consequently, $Y$ is an $M_1$-space.

To complete the proof, denote by $f$ the restriction of the projection map $X \times S \to X$ to $Y$. It is easily seen that $f$ is a continuous mapping from $Y$ onto $X$ and that for each $x \in X$, the set $f^{-1}\{x\}$ is compact in $Y$. We show that the mapping $f$ is closed. Let $C$ be a closed subset of $Y$, and let $x \in X \sim f(C)$. Then

$$C \cap f^{-1}\{x\} = \varnothing$$

and hence $(x, 0) \notin C$. Consequently, there exists a neighborhood $U$ of $x$ in $X$ and $m \in N$ such that $[U \cap \{0\} \cup \{1/n| n > m\}] \cap C = \varnothing$. For each $n < m$, since $(x, 1/n) \notin C$ and since $A_n$ is a closed and discrete subspace of $X$, the set $V_n = X \sim \{a \in A_n | (a, 1/n) \in C\}$ is a neighborhood of $x$ in $X$. It is easily seen that the neighborhood $U \cap \bigcup_{n<m} V_n$ of $x$ is disjoint from the set $f(C)$. We have shown that the mapping $f$ is closed; hence $f$ is a perfect mapping. Setting $g(x) = (x, 0)$ for each $x \in X$, we obtain a homeomorphism between $X$ and the subspace $X' = X \times \{0\}$ of $Y$. That $f$ is a retraction follows by observing that the restriction of the mapping $g \circ f$ to $X'$ is the identity map on $X'$. 
Since stratifiability is a hereditary property [3] and since the continuous image of a stratifiable space under a closed mapping is stratifiable [1], it follows from the above theorem that the problem, whether every stratifiable space is an $M_1$-space, is equivalent with some problems concerning the preservation of the $M_1$-property in topological operations.

**Corollary.** The following statements are mutually equivalent.

(i) Every stratifiable space is an $M_1$-space.

(ii) For every $M_1$-space, every closed subspace of the space is an $M_1$-space.

(iii) For every $M_1$-space, all the images of the space under perfect mappings are $M_1$-spaces.

The implication (ii) $\Rightarrow$ (iii) in the Corollary also follows from the result of C. Borges and D. Lutzer that if every closed subspace of a space is an $M_1$-space, then every image of the space under a perfect mapping is an $M_1$-space [2].

Let us call a topological space an $M_0$-space if the topology of the space has a $\sigma$-closure-preserving base consisting of sets that are both open and closed. Every $M_0$-space is a strongly 0-dimensional $M_1$-space. It is easily seen that a subspace of an $M_0$-space is again an $M_0$-space; from the result of Borges and Lutzer mentioned above it follows that the image of an $M_0$-space under a perfect mapping is an $M_1$-space. It is not known whether every $M_1$-space can be represented as the image of an $M_0$-space under a perfect mapping; for metrizable spaces, such a representation is obtained [7]. Note that it follows from the Theorem that if every $M_1$-space is the image of an $M_0$-space under a perfect mapping, then every stratifiable space is an $M_1$-space.

**References**

5. ______, $\sigma$-discrete stratifiable spaces are $M_1$ (preprint).