PEDERSEN IDEAL AND GROUP ALGEBRAS

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Abstract. For a locally compact $T_2$ group $G$ which has an open subgroup of polynomial growth (e.g., $G$ a group that has a compact neighbourhood invariant under inner automorphisms or $G$ a compact extension of a locally compact nilpotent group) the intersection of the Pedersen ideal of the group $C^*$-algebra with $L^1(G)$ is dense in $L^1(G)$ (Theorem 1). For groups with small invariant neighbourhoods this intersection is the smallest dense ideal of $L^1(G)$, and it consists exactly of those $f \in L^1(G)$ whose "Fourier transform" vanishes outside some (closed) quasicompact subset of $G$ (Theorem 3); the Pedersen ideal of $C^*(G)$ is described as the set of all $a \in C^*(G)$ for which $\{ \pi \in \hat{G} : \pi(a) \neq 0 \}$ is contained in some (closed) quasicompact subset of $\hat{G}$ (Theorem 2).

1. Introduction. In [11] G. K. Pedersen proved that every $C^*$-algebra $A$ has a smallest dense order-related ideal $K_A$, and in 1975 K. B. Laursen and A. M. Sinclair showed that $K_A$ (the so-called Pedersen ideal of $A$) is the smallest ideal among all dense ideals of $A$ [5]. In [12] Pedersen asked whether $K_G \cap L^1(G)$ is dense in $L^1(G)$ where $G$ is a locally compact $T_2$ group, $L^1(G)$ its group algebra and $K_G$ the Pedersen ideal of the group $C^*$-algebra $C^*(G)$. The answer is affirmative in the case where $G$ is abelian or compact (well known) or a connected real nilpotent Lie group [13].

In this note we shall show that the answer is affirmative even in the case where $G$ has at least one compact neighbourhood $U$ of the group identity such that $\lambda(U^k) = O(k^n)$ for some fixed $n \in \mathbb{N}$ ($\lambda$ left Haar-measure). Examples of such groups are, e.g., groups that contain an open subgroup which is a compact extension of a (locally compact) nilpotent group, and IN-groups ($G \in [\text{IN}] \leftrightarrow G$ has a compact invariant neighbourhood); the latter because the open subgroup $G_F$ consisting of all elements with relatively compact conjugacy classes has polynomial growth [10].

In the special case of a SIN-group $G$ ($G \in [\text{SIN}] \leftrightarrow G$ has a fundamental system of compact invariant (under inner automorphisms) neighbourhoods of the identity of $G$) we show that $K_G$ consists exactly of those $a \in C^*(G)$ for which $\{ \pi \in \hat{G} : \pi(a) \neq 0 \}$ is contained in a quasicompact subset of $\hat{G}$, and that $L^1(G) \cap K_G$ is the smallest dense ideal of $L^1(G)$.

The question whether there is a locally compact group $G$ at all for which $L^1(G) \cap K_G$ is not dense in $L^1(G)$ or even $L^1(G) \cap K_G = \{0\}$ still seems to be open.

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REMARK. The existence of a smallest dense ideal in $L^1(G)$, $G \in [\text{SIN}]$, has also been shown in [4].

2. Pedersen ideal and groups of polynomial growth. Let $A$ be a $C^*$-algebra. The Pedersen ideal $K_A$ can be obtained in the following way (see [12]): $K_A$ is the complex linear span of the invariant face generated by the set

$$K_A^+ := \{ x \in A^+: \exists y \in A^+ \text{ with } x = xy\}.$$  

(A face $F$ is a convex cone in $A^+$ such that: $x \in F$, $z \in A^+$, $z < x \Rightarrow z \in F$; $F$ is called invariant if $a^*Fa \subseteq F \forall a \in A \Leftrightarrow u^*Fu = F \forall u \in A$, $u$ unitary, $A$ the $C^*$-algebra obtained by adjunction of a unit (if $A$ does not have a unit)).

For group algebras of groups with polynomial growth, J. Dixmier has found in [1] a functional calculus which turns out to be a very useful tool in harmonic analysis (see e.g. [7] and [8]).

Let $C_n$ ($n \in \mathbb{N}$) denote the set of all functions $\varphi: \mathbb{R} \rightarrow \mathbb{C}$, $\varphi(0) = 0$ which have continuous and integrable derivatives of order $\leq n + 3$. Let $V$ be a compact neighbourhood of the identity of a locally compact group $G$ such that $\lambda(V^k) = O(k^n)$, $f = f^* \in L^1(G) \cap L^2(G)$ such that $f = 0$ outside $V$. Then for every $\varphi \in C_n$ the integral

$$\varphi(f) := \frac{1}{2\pi} \int_{\mathbb{R}} \exp(\lambda f) \hat{\varphi}(\lambda) \, d\lambda$$

converges in $L^1(G)$ ($\hat{\varphi}$ is the Fourier transform of $\varphi$; “exp” is with respect to convolution) and for every *-representation $\pi$ of $L^1(G)$ on a Hilbert space

$$\pi(\varphi(f)) = \varphi(\pi(f))$$

where the right side is defined by the usual operational calculus on the hermitian operator $\pi(f)$.

THEOREM 1. Let $G$ be a locally compact $T_2$ group with a compact neighbourhood $V$ of the identity $e$ of $G$ such that $\lambda(V^k) = O(k^n)$ for some $n \in \mathbb{N}$ (equivalently: $G$ contains an open subgroup of polynomial growth). Then $L^1(G) \cap K_G$ is a dense ideal in $L^1(G)$ (where $K_G$ denotes the Pedersen ideal of $C^*(G)$).

PROOF. Take a compact neighbourhood $U = U^{-1}$ of $e$ such that $U^{2(n+4)} \subseteq V$, $(g_i)_{i \in I}$ a bounded approximate unit for $L^1(G)$, $g_i: G \rightarrow \mathbb{R}$ continuous, $g_i(x) > 0$ $\forall x \in G$, supp$(g_i) \subseteq U$, $\|g_i\|_1 = 1$, and define $f_i := g_i^* \cdot g_i$. Take $\varphi \in C_n$ such that

$$\varphi(t) = t^{n+4}$$

for all $t$ with $|t| < 1 = \|f_i\|_1$, $\varphi(t) > 0 \ \forall t > 0$.

Now fix $i$, choose $\varepsilon > 0$. There is a real valued $\psi_{i,\varepsilon} \in C_n$ with

$$\psi_{i,\varepsilon}(t) = \varphi(t) \ \forall t \in \mathbb{R} \setminus [-1, +1],$$

$$|\psi_{i,\varepsilon}^{(\alpha)}(t) - \varphi^{(\alpha)}(t)| < \varepsilon/A_i \ \forall t \in [-1, +1], \alpha = 0, 1, \ldots, n + 3,$$

and

$$\psi_{i,\varepsilon}(t) = 0 \ \forall |t| < \delta_{i,\varepsilon} \text{ for some } \delta_{i,\varepsilon} \in (0, 1).$$

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[1, Lemme 8] where $A_i$ is the constant $A$ (independent of $e$) in the proof of [1, Théorème 1.b]. Without loss of generality $\psi_{t, \varepsilon}(t) > 0 \, \forall t > 0$.

Now we have $\varphi(f_t) = f_t^{*+\delta}$ (exponent with respect to convolution) and

$$\|\psi_{t, \varepsilon}(f_t) - f_t^{*+\delta}\|_1 < \varepsilon$$

[1, Théorème 1.b], hence $\{\psi_{t, \varepsilon}(f_t) : i \in I, 0 < \varepsilon < 1\}$ is a bounded approximating set for $L^1(G)$. If we can show all $\psi_{t, \varepsilon}(f_t)$ to be in $K_G$ the proof is finished, since $\{f \ast \psi_{t, \varepsilon}(f_t) : f \in L^1(G), i \in I, 0 < \varepsilon < 1\}$ is dense in $L^1(G)$.

By construction $f_t \in C^*(G)^+$, and its spectrum $\sigma_{C^*(G)}(f_t)$ is contained in $[0, 1]$. Since $\psi_{t, \varepsilon}(f_t) = \psi_{t, \varepsilon}(f_t)$ (take for $\pi$ the universal representation in ($\ast$); $\psi_{t, \varepsilon}(f_t)$ means the usual functional calculus for $C^*$-algebras) and $\psi_{t, \varepsilon}(t) > 0 \, \forall t > 0$, we have $\psi_{t, \varepsilon}(f_t) \in C^*(G)^+$. Choose a function $\rho_{t, \varepsilon} : \mathbb{R} \to [0, \infty)$, $\rho_{t, \varepsilon} \in C_n$ such that

$$\rho_{t, \varepsilon}(t) = 1 \quad \forall t \in [\delta_{t, \varepsilon}, 1]$$

and

$$\rho_{t, \varepsilon}(t) = 0 \quad \forall t \in \mathbb{R} \setminus [1/2 \cdot \delta_{t, \varepsilon}, 2].$$

Now $\rho_{t, \varepsilon}(f_t) \in L^1(G)$, $\rho_{t, \varepsilon}(f_t) = \rho_{t, \varepsilon}(f_t) \in C^*(G)^+$ and $\psi_{t, \varepsilon}(f_t) \rho_{t, \varepsilon}(f_t) = \psi_{t, \varepsilon}(f_t)$, hence $\psi_{t, \varepsilon}(f_t) \in K^{-\infty} \subseteq K_G$.

**Remark 1.** The properties of the bounded approximating set in the proof of Theorem 1 show at once that $I \cap K_G$ is dense in $I$ for every left (or right) ideal in $L^1(G)$ ($I$ not necessarily closed).

**Remark 2.** The construction of the Pedersen ideal $K_A$ of a $C^*$-algebra $A$ shows at once that for all elements $x \in K_A$ the set $\{\pi \in \hat{A} : \pi(x) \neq 0\}$ is contained in a quasicompact subset of $\hat{A}$ since $ab = b \, (a, b \in A^+)$ implies $\{\pi \in \hat{A} : \pi(b) \neq 0\} \subseteq \{\pi \in \hat{A} : \|\pi(a)\| > 1\}$. Hence we have the following

**Corollary.** $G$ as in Theorem 1. The set of all $f \in L^1(G)$ for which the “Fourier transform” $\hat{f}, \hat{f}(\pi) : = \pi(f), \pi \in \hat{G}$, vanishes outside a quasicompact set of $\hat{G}$ is dense in $L^1(G)$.

3. **Pedersen ideal and SIN-groups.** For SIN-groups we get more detailed information than in the corollary above:

**Theorem 2.** Let $G \in [SIN], K_G$ the Pedersen ideal of $C^*(G)$, $J_G := \{a \in C^*(G); \hat{a}$ vanishes outside a quasicompact subset of $\hat{G}\}$ (where $\hat{a}(\pi) := \pi(a) \, \forall \pi \in \hat{G}$). Then $J_G = K_G$.

**Proof.** We only have to show $J_G \subseteq K_G$. Consider the following mappings $t$ and $p$:

$$t : \hat{G} \to \text{Prim} C^*(G), \quad \pi \mapsto \ker \pi,$$

$$p : \text{Prim} C^*(G) \to \text{G-Max} C^*(G_F) \cong E(G_F, G), \quad P \mapsto P \cap C^*(G_F),$$

where $\text{G-Max} C^*(G_F)$ denotes the ideals of $C^*(G_F)$ which are maximal among the $G$-invariant modular ideals; $\text{G-Max} C^*(G_F)$ with hull-kernel topology is homeomorphic to $E(G_F, G)$ (the extreme points of the set of all $G$-invariant continuous positive definite functions $\gamma$ on $G_F$ with $\gamma(e) = 1$) with the topology of compact
convergence; the homeomorphism
\[ E(G_F, G) 	o G\text{-Max } C^*(G_F) \]
is given by
\[ \gamma \mapsto \{ a \in C^*(G_F) : \langle a^*a, \gamma \rangle = 0 \} \quad [9, (4)]. \]

For each \( P \in \text{Prim } C^*(G) \) there is a continuous positive definite indecomposable function \( \varphi, \varphi(e) = 1 \) with \( P = \ker \pi_\varphi \), and \( P \cap C^*(G_F) \) corresponds to \( (\varphi|G_F)^G \in E(G_F, G) \), which is defined by
\[ (\varphi|G_F)^G(n) := \int_{\overline{I(G_F,G)}} \varphi(\beta^{-1}(n)) \, d\beta \]
where \( \overline{I(G_F,G)} \) is a compact group: the closure of the restrictions to \( G_F \) of the inner automorphisms of \( G \). The mapping \( \pi : \ker \pi_\varphi \mapsto (\varphi|G_F)^G \) is well-defined from \( \text{Prim } C^*(G) \) onto \( E(G_F, G) \) (even continuous and proper). See [2].

Now take an arbitrary \( a \in J_G \), choose \( L \subseteq \text{Prim } C^*(G) \) quasicompact with \( t^{-1}(L) \supseteq \{ \pi \in \hat{G} : \pi(a) \neq 0 \} \). Since the “Fourier transform” of the hermitian and positive parts of \( a \) vanish outside \( t^{-1}(L) \) too, \( a > 0 \) without loss of generality. Since the algebra of functions in \( L^1(G_F) \) that are central in \( L^1(G) \) is a completely regular Banach algebra with maximal ideal space \( E(G_F, G) \) (see [3, (4)] or [6, (2.4)]) we can get \( f \in L^1(G_F) \), central in \( L^1(G) \), with
\[ \hat{f}(\alpha) := \int_{G_F} f(x)\alpha(x) \, dx > 0 \quad \forall \alpha \in E(G_F, G), \]
\[ \hat{f}(\alpha) = 1 \quad \forall \alpha \in p(L) \subseteq E(G_F, G). \]
Then \( f \in C^*(G_F)^+ \subseteq C^*(G)^+ \) and \( \pi(f) = \text{id}_{H_\varphi} \forall \pi \in t^{-1}(L) \subseteq \hat{G} \), hence \( fa = a \), hence \( a \in K^\infty_G \subseteq K_G \).

Let us check \( \pi(f) = \text{id}_{H_\varphi} \forall \pi \in t^{-1}(L) \). Let \( \pi \in \hat{G}, \varphi \) with \( \pi_\varphi = \pi \):
\[ \hat{f}((\varphi|G_F)^G) = \int_{G_F} f(x)(\varphi|G_F)^G(x) \, dx = \int_{G_F} f^G(x)\varphi(x) \, dx \]
\[ = \int_{G} f(x)\varphi(x) \, dx = \int_{G} f(x)(\pi(x)\xi_\varphi, \xi_\varphi) \, dx \]
\[ = (\pi(f)\xi_\varphi, \xi_\varphi). \]
Since \( f \) is central and \( \pi \) irreducible, \( \pi(f) \) is a multiple of \( \text{id}_{H_\varphi} \), so we have
\[ \pi(f) = \hat{f}((\varphi|G_F)^G) \cdot \text{id}_{H_\varphi} \forall \pi \in \hat{G}, \varphi \text{ with } \pi_\varphi = \pi; \]
hence the assertion.

Remark. In a SIN-group \( G \) each quasicompact subset of \( \hat{G} \) is contained in a closed quasicompact subset of \( \hat{G} \) (because the mapping \( p \) in the proof of Theorem 2 is continuous and proper).

Lemma. \( G \in \text{[SIN]}, I \) a dense ideal in \( L^1(G) \). Then for every quasicompact set \( L \subseteq \text{Prim } C^*(G) \) there is a \( u \in I \) such that \( u \) is a unit for \( L^1(G) \) modulo \( k(L) \cap L^1(G) \). \[ k(L) := \cap \{ P \in \text{Prim } C^*(G) : P \in L \} \].
Proof. For $L$ quasicompact take $f$ as in the proof of Theorem 2; then $f$ is a unit for $L_1(G)/(k(L) \cap L_1(G))$. Since $k(L) \cap L_1(G)$ is a modular ideal in $L_1(G)$, $I + (k(L) \cap L_1(G)) = L_1(G)$ [14, 2.6.8] and thus there is a $d \in k(L) \cap L_1(G)$ and $u \in I$ with $d + u = f$, hence $u - f \in k(L) \cap L_1(G)$ and thus $u$ is a unit for $L_1(G)$ modulo $(k(L) \cap L_1(G))$.

**Theorem 3.** Let $G \in [SIN]$. Then there is a smallest dense ideal $I$ in $L_1(G)$. $I$ coincides with the intersection of $L_1(G)$ and the Pedersen ideal $K_G$ of $C^*(G)$ and also with the set of all $h \in L_1(G)$ for which the “Fourier transform” $\hat{h}$ vanishes outside a quasicompact subset of $\hat{G}$.

Proof. $I := L_1(G) \cap K_G$ is dense in $L_1(G)$. Now let $I$ be an arbitrary dense ideal in $L_1(G)$. For every $a \in I$ there exists a quasicompact set $L \subseteq \text{Prim } C^*(G)$ with $L \supseteq \{ P \in \text{Prim } C^*(G); a \not\in P \}$, hence by the lemma above there is a $u \in I$ with $ua + P = a + P \forall P \in L$, and for all $P \in \text{Prim } C^*(G) \setminus L$ too (since then $a \in P$). Thus we have $ua = a$, hence $a \in I$.

The last assertion follows from Theorem 2.

4. Added in proof. 1. Let $G = G_{a,b}(0)$ be the group of all $3 \times 3$ matrices of the form

$$\begin{bmatrix}
1 & x & z \\
0 & e^r & y \\
0 & 0 & 1
\end{bmatrix}, \quad r, x, y, z \in \mathbb{R}.$$ 

Then the intersection of all dense two-sided ideal in $L_1(G)$ is trivial. The intersection of the Pedersen ideal of $C^*(G)$ with $L_1(G)$ is trivial, too. (Communicated by Viktor Losert.)

2. Let $G$ be a locally compact $T_2$ group. If $G_0 \in [IN]$ ($G_0$ the identity component) or if $G$ is a group of polynomial growth with symmetric group algebra $L_1(G)$, then there does exist a smallest dense two-sided ideal of $L_1(G)$ (V. Losert, resp., J. Ludwig).

**References**


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