ON ATRIODIC TREE-LIKE CONTINUA

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Abstract. D. P. Bellamy has recently shown that atriodic tree-like continua do not have the fixed point property for homeomorphisms. J. B. Fugate and T. B. McLean showed that hereditarily indecomposable tree-like continua have the fixed point property for pointwise periodic homeomorphisms. In this paper the latter result is extended to the case of atriodic tree-like continua. In the course of the proof it is shown that the property of being an atriodic tree-like continuum is a Whitney property. In particular, it is shown that the hyperspace of an atriodic tree-like continuum is at most 2-dimensional.

1. Introduction. A continuum is a compact, connected, metric space. A tree is a finite, connected, simply connected, one-dimensional polyhedron. A continuum is tree-like if it admits finite open covers of arbitrarily small mesh whose nerves are trees. A continuum $X$ is said to be a triod (resp. $n$-od) if there exists a subcontinuum $M$ of $X$ such that $X \setminus M$ has at least three (resp. at least $n$) components. We say $X$ is atriodic if $X$ contains no triod. A continuum is hereditarily indecomposable if and only if it contains no 2-od.

If $X$ is a continuum we let $C(X)$ denote the hyperspace of subcontinua of $X$ with the Hausdorff metric. A Whitney map for $X$ is a mapping $\mu : C(X) \to [0, \infty)$ such that $\mu(\{x\}) = 0$ for each $x \in X$ and $\mu(A) < \mu(B)$ for each $A, B \in C(X)$ with $A \subsetneq B$. A Whitney level for $X$ is a set $\mu^{-1}(t)$ where $0 < t < \mu(X)$. Whitney levels are continua in $C(X)$ (see [9, p. 400]). The existence of Whitney maps for $X$ is well known (see [9]).

2. Whitney property. A property $P$ of continua is said to be a Whitney property if whenever $A$ is a continuum with property $P$ then every Whitney level of $A$ also has property $P$. Krasinkiewicz in [6] and [7] proved that being an arc-like continuum, being a proper circle-like continuum or being an hereditarily indecomposable tree-like continuum is a Whitney property. The main purpose of this section is to show that being an atriodic tree-like continuum is also a Whitney property. This provides a converse to a result of Nadler [8, 3.5] who has shown that if $X$ is a continuum whose Whitney levels are tree-like then $X$ is atriodic and tree-like.

A continuum $X$ is said to have the covering property (see [9]) if for each Whitney level $\mu^{-1}(t)$ of $X$ and each subcontinuum $\Lambda$ of $\mu^{-1}(t)$, $\bigcup \Lambda = X$ implies $\Lambda = \mu^{-1}(t)$.

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Theorem 2.1 (See [9, p. 485]). A continuum $X$ has the covering property if and only if each Whitney level of $X$ is irreducible.

Theorem 2.2 [4, 5.6]. Atriodic tree-like continua have the covering property.

Lemma 2.3. Let $X$ be an atriodic tree-like continuum. Let $p \in X$ and let $\mu$ be a Whitney map for $X$. If $\Lambda$ is a subcontinuum of $\mu^{-1}(t)$ for some Whitney level then $K = \Lambda \cap \{A \in \mu^{-1}(t) \mid p \in A\}$ is an arc or a point or $K$ is empty.

Proof. We suppose $p \in \bigcup \Lambda$. Since $\bigcup \Lambda$ is a continuum, it has the covering property by Theorem 2.2. Now $\mu|_{\bigcup \Lambda}$ is a Whitney map for $\bigcup \Lambda$ and hence $\Lambda$ is a Whitney level of $\bigcup \Lambda$. By [9, p. 405] $\{A \in \Lambda \mid p \in A\} = K$ is an arcwise connected continuum.

Now $\bigcup K$ is an atriodic tree-like continuum. By Theorem 2.2 $\bigcup K$ has the covering property. Hence, $K$ is a Whitney level of $\bigcup K$. By Theorem 2.1 $K$ is irreducible.

Sorgenfrey proved in [13, Theorem 1.8] that if $T$ is the union of three continua which have a point in common and such that no one of them is a subset of the union of the other two then $T$ contains a triod.

Theorem 2.4. The property of being an atriodic tree-like continuum is a Whitney property.

Proof. Let $X$ be an atriodic tree-like continuum and let $\mu^{-1}(t)$ be a Whitney level of $X$. If $\Lambda$ is a subcontinuum of $\mu^{-1}(t)$ then as in the proof of Lemma 2.3 $A$ is a Whitney level of the tree-like continuum $\bigcup \Lambda$. By [11, Theorem 5] the first Čech cohomology group of $\Lambda$ is trivial.

It follows that $\dim \mu^{-1}(t) = 1$. For if $Y$ is a continuum with $\dim Y > 2$ then there exists an essential map $f$ of $Y$ onto $B$, the closed unit disk in the plane [10, p. 127]. Then $f|_{f^{-1}(S^1)}$ is essential where $S^1$ is the boundary of $B$ and, hence, $f$ is essential on some subcontinuum $Z$ of $f^{-1}(S^1)$, i.e. $H^1(Z) \neq 0$.

Let $K = \{(A, x) \mid x \in A \in \mu^{-1}(t)\} \subset \mu^{-1}(t) \times X$. Then $K$ is a continuum and $\dim K < 2$. Let $\pi_1: K \to \mu^{-1}(t)$ and $\pi_2: K \to X$ be the coordinate projections. The point inverses under $\pi_1$ are tree-like continua and the point inverses under $\pi_2$ are arcs or points by Lemma 2.3. In particular, the point inverses of $\pi_1$ and $\pi_2$ have trivial shape. By a theorem of Sher [12] $\mu^{-1}(t)$ and $X$ have the same shape since $\pi_1$ and $\pi_2$ are cell-like mappings between finite dimensional spaces. By the theorem of Case and Chamberlin [2] $\mu^{-1}(t)$ is tree-like.

By Theorem 2.2 $X$ has the covering property hereditarily. Therefore, by [9, p. 510] $\mu^{-1}(t)$ is hereditarily irreducible and, hence, $\mu^{-1}(t)$ is atriodic. This completes the proof of the theorem.

Corollary 2.5. If $X$ is an atriodic tree-like continuum, then $C(X)$ is 2-dimensional.

3. The fixed point theorem. In [3] Fugate and McLean proved the following two results.

Theorem 3.1 [3, 1.5]. Tree-like continua have the fixed point property for periodic homeomorphisms.
Theorem 3.2 [3, 3.3]. Hereditarily indecomposable tree-like continua have the fixed point property for pointwise periodic homeomorphisms.

In this section we extend Theorem 3.2 to the case of atriodic tree-like continua. In our argument we use Theorem 2.4 and follow the argument given in [3]. First we prove the following lemma.

Lemma 3.3. If \( M \) is an atriodic, hereditarily unicoherent continuum and if \( h: M \to M \) is a pointwise periodic homeomorphism, then the induced homeomorphism \( \tilde{h}: C(M) \to C(M) \) which is defined by \( \tilde{h}(Y) = h(Y) \) for each \( Y \in C(M) \) is pointwise periodic. Moreover, if \( x \in A \subset C(M) \) and \( h^n(x) = x \), then \( \tilde{h}^2n(A) = A \).

Proof. Let \( x \in A \subset C(M) \) and suppose \( h^n(x) = x \). If \( y \in A \) then there is a unique continuum \( B_y \) in \( M \) which is irreducible from \( x \) to \( y \) since \( M \) is hereditarily unicoherent. Then \( B_y \subset A \) and \( A = \bigcup \{ B_y \mid y \in A \} \). Let \( y \in A \setminus \{ x \} \) and let \( B = B_y \). It suffices to show that \( h^n(B) = B \). Suppose \( h^n(B) \neq B \). If \( B \subsetneq h^n(B) \) let \( z \in h^n(B) \setminus B \). Then \( h^n(z) \in h^{i+1}(B) \setminus h^i(B) \) for each \( i \). Since \( h^n(B) \subset h^{i+1}(B) \) for each positive integer \( i \) this would imply \( z \) has infinite order under \( h \) and so would contradict the pointwise periodicity of \( h \). Thus, \( B \subsetneq h^n(B) \). Similarly, \( h^n(B) \subsetneq B \). Notice that \( h^n(y) \notin B \) since \( B \cap h^n(B) \) is a proper subcontinuum of \( h^n(B) \) and \( h^n(B) \) is irreducible between \( h^n(x) = x \) and \( h^n(y) \). Similarly, \( h^n(y) \notin h^{2n}(B) \), \( h^{2n}(y) \subset B \cup h^n(B) \) and \( y \notin h^n(B) \cup h^{2n}(B) \). By Sorgenfrey's theorem [13, 1.8] \( M \) contains a triod. This is a contradiction. Thus, we have proved \( B = h^{2n}(B) \) and, hence, \( A = \tilde{h}^{2n}(A) \).

Theorem 3.4. Suppose \( M \) is an atriodic tree-like continuum and \( h: M \to M \) is a pointwise periodic homeomorphism. Then \( h \) has a fixed point.

Proof. Suppose \( h \) does not have a fixed point. We may suppose \( M \) is minimal with respect to being mapped into itself. Hence, \( M \) is not a point and if \( Y \) is a proper subcontinuum of \( M \), \( h(Y) \not\subset Y \).

Let \( \mu \) be a Whitney map for \( M \). We may suppose \( \mu(M) = 1 \). Let \( \hat{h}: C(M) \to C(M) \) be the map induced by \( h \).

For \( x \in M \) let \( O(x) = \min_{n>0} (h^n(x) = x) \). Let \( J_i = \{ x \in M \mid O(x) < i \} \). Then \( J_i \) is closed and \( M = J_1 \cup J_2 \cup \ldots \). By the Baire Category Theorem there exists \( n \) such that \( J_n \) has nonvoid interior in \( M \). From the above it follows that there exists \( s \) with \( 0 < s < 1 \) such that if \( K \in \mu^{-1}(s, 1] \) then \( K \cap J_n \neq \emptyset \).

Define \( \sigma: C(M) \to [0, 1] \) by

\[
\sigma(A) = \max \{ \mu(\tilde{h}(A)) \mid 1 < i < 2n! \}.
\]

Then \( \sigma \) is clearly a Whitney map for \( M \) such that if \( \mu(A) > s \) then \( \sigma(\tilde{h}(A)) = \sigma(A) \). Since \( \mu^{-1}(1) = \sigma^{-1}(1) = \{ M \} \) there exists \( 0 < t < 1 \) such that \( \sigma(A) > t \) implies \( \mu(A) > s \).

By Theorem 2.4 \( \sigma^{-1}(t) \) is a tree-like continuum in \( C(M) \). The restriction of \( \tilde{h} \) to \( \sigma^{-1}(t) \) is a periodic homeomorphism of \( \sigma^{-1}(t) \) (of period \( < 2n! \)). By Theorem 3.1 \( \tilde{h}(A) = A \) for some \( A \in \sigma^{-1}(t) \). This contradicts the assumption at the beginning of the proof that \( \tilde{h}(Y) \not\subset Y \) for each proper subcontinuum \( Y \) of \( M \).
Question. Is Theorem 3.4 true for tree-like continua which do not contain \( n \)-ods for arbitrarily large \( n \)?

References


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