A SIMPLIFICATION OF THE COMPUTATION OF THE
NATURAL REPRESENTATION OF THE SYMMETRIC GROUP $S_n$

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Abstract. Recent use of the symmetric group $S_n$ in processing identities in
nonassociative algebras has brought a renewed interest in the natural (integral)
irreducible representation of $S_n$ [3]. Using a construction due to A. Young, H.
Boerner gives a prescription for writing down the natural (integral) irreducible
representation of the symmetric group $S_n$ over an algebraically closed field of
characteristic zero [1, p. 119]. Writing down the matrices using this prescription is
rather tedious, and becomes computationally impossible for large $n$ ($n > 9$) be-
cause of the need to calculate chains. In this paper, we simplify the computation of
these matrices by eliminating the need to calculate chains. The calculation of the
chains is replaced by the simple act of setting up and inverting an upper triangular
matrix $A_f$.

Let $n$ be a fixed positive integer. By a frame, we will mean $n$ squares arranged in
rows of decreasing length such that the left-hand edges of the rows coincide. A
tableau is obtained by placing the numbers 1 through $n$ in the squares of a frame.
For a given frame, let $[\pi]$ denote the matrix corresponding to the natural (integral)
irreducible representation (over an algebraically closed field of characteristic zero)
of $\pi \in S_n$. For each $\pi \in S_n$, we show how to construct a matrix $A_\pi$ such that
$[\pi] = A_I^{-1}A_\pi$, where $I$ is the identity permutation in $S_n$.

Throughout the remainder of this paper, we will assume that all tableaux belong
to a given frame of $n$ squares. We will also assume that the irreducible representa-
tion of $S_n$ corresponding to this frame is over $F$, an algebraically closed field of
characteristic zero, and that the degree of this representation is $f$.

Given a tableau $T$, let $P$ denote the product of the positive symmetric groups of
the rows of $T$, and let $N$ denote the product of the negative symmetric groups of
the columns of $T$ [5, p. 391]. Let $e = f/n!PN$. Then $e$ is a nonzero idempotent in
the group algebra of $S_n$ over $F$ [1, p. 109], which we will call the idempotent
Corresponding to $T$.

Let $T_1, T_2, \ldots, T_n$, be the distinct tableaux for our given frame, and let
$e_1, e_2, \ldots, e_n$ be the corresponding idempotents. For $i, j = 1, 2, \ldots, n!$, let $s_{ij}$
denote the permutation in $S_n$ which transforms $T_j$ into $T_i$. It is clear that $s_{ij}^{-1} = s_{ji}$
and $s_{ij}s_{jk} = s_{ik}$. Also, $e_i = s_{ij}e_je_{ji}$ [1, p. 105].

For $i, j = 1, 2, \ldots, n!$, we define numbers $e_{ij}$ as follows:

If there exists two numbers in one column of $T_i$ and in one row of $T_j$, then we set $e_{ij} = 0$. If not, then there exists a vertical permuta-
tion \( q \) for \( T_i \) which leaves the columns of \( T_i \) fixed as sets and takes the numbers of \( T_j \) into the correct rows they occupy in \( T_j \) \([1, \text{pp. } 106-108]\). In this case, we set \( e_y = \text{sgn}(q) \).

Then we have \( e_i e_j = e_y s_y e_j \) for \( i, j = 1, 2, \ldots, n! \) \([4, \text{p. } 22]\).

Let \( \pi \in S_n \) and \( 1 < i, j < n! \). Then \( \pi T_j = T_r \) for some \( r \), so \( \pi = s_{ij} \). Therefore, \( e_i \pi e_j = e_i s_y e_j = e_r e_r e_s e_s = e_y s_y e_j \). Denote \( e_r \) by \( e_y \). Then \( e_i \pi e_j = e_y s_y e_j \) for each \( \pi \in S_n \) and for each \( i, j = 1, 2, \ldots, n! \).

Let \( T_1, T_2, \ldots, T_f \) denote the standard tableaux in dictionary order \([1, \text{p. } 112]\).

For each \( \pi \in S_n \), form the \( f \times f \) matrix \( A^\pi \) by letting \( [A^\pi]_{ij} = e_i e_j \) for \( i, j = 1, 2, \ldots, f \). Thus, the \( i, j \) entry of \( A^\pi \) can be calculated as follows:

Apply \( \pi \) to the tableau \( T_j \). If there exist two numbers that appear together in a column of \( T_i \) and a row of \( \pi T_j \), then \([A^\pi]_{ij} = 0\). If not, then \([A^\pi]_{ij}\) equals the sign of the vertical permutation for \( T_i \) which leaves the columns of \( T_i \) fixed as sets and takes the numbers of \( T_i \) into the correct rows they occupy in \( \pi T_j \).

**Theorem.** For each \( \pi \in S_n \), \([\pi] = A^{-1}, A^\pi\), where \( I \) is the identity permutation of \( S_n \).

**Proof.** \( A^\pi \) is upper triangular with \([A^\pi]_{ii} = +1\) for \( i = 1, 2, \ldots, f \) \([1, \text{pp. } 109 \text{ and } 113]\). Thus, \( A^\pi \) has determinant \( +1 \), and hence is invertible. Let \( w_y = [A^{-1}]_{ij} \), \( i, j = 1, \ldots, f \). If we denote the natural units \([4, \text{p. } 51]\) by \( g_y \), then \( g_y = \sum_{k=1}^{f} s_k e_k w_{yk} \) \([4, \text{p. } 53]\), and \( g_y g_{km} = \delta_{jk} g_{im} \) \([4, \text{p. } 53]\). Now \([\pi] g_y = g_i \pi g_{ji} \) \([1, \text{p. } 117]\). We compute

\[
[g_i \pi g_{ji}]_{ij} = \left( \sum_{k=1}^{f} s_k e_k w_{ik} \right) \pi \left( \sum_{m=1}^{f} s_m e_m w_{jm} \right) = \sum_{k,m=1}^{f} w_k e_k s_k e_j s_m w_{jm} = \left( \sum_{k=1}^{f} w_k e_k \right) g_{ij}.
\]

Therefore, \([\pi]_{ij} = \sum_{k=1}^{f} w_k e_k \). Thus, \([\pi] = A^{-1} A^\pi\).

Note that we get an equivalent representation of \( S_n \) for any set of \( f \) tableaux (not necessarily standard) such that \( A^\pi \) is invertible. If the determinant of \( A^\pi \) is \( +1 \) or \(-1\), then the representation of \( S_n \) will be integral. In \([2]\), the possibility of choosing \( f \) tableaux (not necessarily standard) such that \( A^\pi \) is the identity matrix is considered. For frames which this is possible, we have that \( \{ A^\pi : \pi \in S_n \} \) is an easily computed irreducible representation of \( S_n \) in which the entries of all the matrices are restricted to \( +1, -1, \) or \( 0 \). Other advantages to such a set of tableaux are listed in \([2]\).

As a final note, we mention that the first footnote in \([4, \text{p. } 53]\) should read “*Young IV, p. 258*”. Also, the “r” and “s” are interchanged in the next to last line in \([5, \text{p. } 401]\). Finally, the second and third lines in \([5, \text{p. } 402]\) (Young’s fourth paper
On quantitative substitutional analysis) are in error. To see this, consider the frame (3, 2). Let \( X_{rs} = \delta_{js} \delta_{is} = Y_{rs}, \ r, s = 1, 2, \ldots, 5. \) Since \( M_5 = I, \sum X_{rs} P_\sigma N_s M_5 = \sum Y_{rs} P_\sigma N_s. \) Now \( E - S = A_1^T. \) Thus \( [X_{rs}] = [X_{rs}] \neq [X_{rs}](E - S). \) One can show that the third line in [5, p. 402] should real \( [X_{rs}] = [Y_{rs}](E - S)^T. \)

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References


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