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A SIMPLIFICATION OF THE COMPUTATION OF THE NATURAL REPRESENTATION OF THE SYMMETRIC GROUP S_n

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ABSTRACT. Recent use of the symmetric group S_n in processing identities in nonassociative algebras has brought a renewed interest in the natural (integral) irreducible representation of S_n [3]. Using a construction due to A. Young, H. Boerner gives a prescription for writing down the natural (integral) irreducible representation of the symmetric group S_n over an algebraically closed field of characteristic zero [1, p. 119]. Writing down the matrices using this prescription is rather tedious, and becomes computationally impossible for large n ($n > 9$) because of the need to calculate chains. In this paper, we simplify the computation of these matrices by eliminating the need to calculate chains. The calculation of the chains is replaced by the simple act of setting up and inverting an upper triangular matrix A_I.

Let n be a fixed positive integer. By a frame, we will mean n squares arranged in rows of decreasing length such that the left-hand edges of the rows coincide. A tableau is obtained by placing the numbers 1 through n in the squares of a frame. For a given frame, let $[\pi]$ denote the matrix corresponding to the natural (integral) irreducible representation (over an algebraically closed field of characteristic zero) of $\pi \in S_n$. For each $\pi \in S_n$, we show how to construct a matrix A_π such that $[\pi] = A_I^{-1}A_\pi$, where I is the identity permutation in S_n.

Throughout the remainder of this paper, we will assume that all tableaux belong to a given frame of n squares. We will also assume that the irreducible representation of S_n corresponding to this frame is over F, an algebraically closed field of characteristic zero, and that the degree of this representation is f.

Given a tableau T, let P denote the product of the positive symmetric groups of the rows of T, and let N denote the product of the negative symmetric groups of the columns of T [5, p. 391]. Let $e = f/n!PN$. Then e is a nonzero idempotent in the group algebra of S_n over F [1, p. 109], which we will call the idempotent corresponding to T.

Let $T_1, T_2, \ldots, T_{n!}$ be the distinct tableaux for our given frame, and let $e_1, e_2, \ldots, e_{n!}$ be the corresponding idempotents. For $i, j = 1, 2, \ldots, n!$, let s_{ij} denote the permutation in S_n which transforms T_j into T_i. It is clear that $s_{ij}^{-1} = s_{ji}$ and $s_{ij}s_{jk} = s_{ik}$. Also, $e_i = s_{ij}e_js_{ji}$ [1, p. 105].

For $i, j = 1, 2, \ldots, n!$, we define numbers ε_{ij} as follows:

> If there exists two numbers in one column of T_i and in one row of T_j, then we set $\varepsilon_{ij} = 0$. If not, then there exists a vertical permuta-

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tion q for T_i which leaves the columns of T_i fixed as sets and takes the numbers of T_i into the correct rows they occupy in T_j [**1**, pp. 106–108]. In this case, we set $\varepsilon_{ij} = \operatorname{sgn}(q)$.

Then we have $e_i e_j = \varepsilon_{ij} s_{ij} e_j$ for $i, j = 1, 2, \ldots, n!$ [**4**, p. 22].

Let $\pi \in S_n$ and $1 \leqslant i, j \leqslant n!$. Then $\pi T_j = T_r$ for some r, so $\pi = s_{rj}$. Therefore, $e_i \pi e_j = e_i s_{rj} e_j = e_i e_r s_{rj} = \varepsilon_{ir} s_{ir} e_r s_{rj} = \varepsilon_{ir} s_{ij} e_j$. Denote ε_{ir} by ε_{ij}^{π}. Then $e_i \pi e_j = \varepsilon_{ij}^{\pi} s_{ij} e_j$ for each $\pi \in S_n$ and for each $i, j = 1, 2, \ldots, n!$.

Let $T_1, T_2, \ldots, T_f$ denote the standard tableaux in dictionary order [**1**, p. 112]. For each $\pi \in S_n$, form the $f \times f$ matrix A_π by letting $[A_\pi]_{ij} = \varepsilon_{ij}^{\pi}$ for $i, j = 1, 2, \ldots, f$. Thus, the i, j entry of A_π can be calculated as follows:

Apply π to the tableau T_j. If there exist two numbers that appear together in a column of T_i and a row of πT_j, then $[A_\pi]_{ij} = 0$. If not, then $[A_\pi]_{ij}$ equals the sign of the vertical permutation for T_i which leaves the columns of T_i fixed as sets and takes the numbers of T_i into the correct rows they occupy in πT_j.

THEOREM. *For each $\pi \in S_n$, $[\pi] = A_I^{-1} A_\pi$, where I is the identity permutation of S_n.*

PROOF. A_I is upper triangular with $[A_I]_{ii} = +1$ for $i = 1, 2, \ldots, f$ [**1**, pp. 109 and 113]. Thus, A_I has determinant $+1$, and hence is invertible. Let $w_{ij} = [A_I^{-1}]_{ij}$, $i, j = 1, \ldots, f$. If we denote the natural units [**4**, p. 51] by g_{ij}, then $g_{ij} = \sum_{k=1}^{f} s_{ik} e_k w_{jk}$ [**4**, p. 53], and $g_{ij} g_{km} = \delta_{jk} g_{im}$ [**4**, p. 53]. Now $[\pi]_{ij} g_{ij} = g_{ii} \pi g_{jj}$ [**1**, p. 117]. We compute

$$\begin{aligned} g_{ii}\pi g_{jj} &= \left(\sum_{k=1}^{f} s_{ik} e_k w_{ik}\right)\pi\left(\sum_{m=1}^{f} s_{jm} e_m w_{jm}\right) \\ &= \sum_{k,m=1}^{f} w_{ik} s_{ik} e_k \pi e_j s_{jm} w_{jm} = \sum_{k,m=1}^{f} w_{ik} s_{ik} \varepsilon_{kj}^{\pi} s_{kj} e_j s_{jm} w_{jm} \\ &= \left(\sum_{k=1}^{f} w_{ik} \varepsilon_{kj}^{\pi}\right)\left(\sum_{m=1}^{f} s_{im} e_m w_{jm}\right) = \left(\sum_{k=1}^{f} w_{ik} \varepsilon_{kj}^{\pi}\right) g_{ij}. \end{aligned}$$

Therefore, $[\pi]_{ij} = \sum_{k=1}^{f} w_{ik} \varepsilon_{kj}^{\pi}$. Thus, $[\pi] = A_I^{-1} A_\pi$.

Note that we get an equivalent representation of S_n for any set of f tableaux (not necessarily standard) such that A_I is invertible. If the determinant of A_I is $+1$ or -1, then the representation of S_n will be integral. In [**2**], the possibility of choosing f tableaux (not necessarily standard) such that A_I is the identity matrix is considered. For frames which this is possible, we have that $\{A_\pi : \pi \in S_n\}$ is an easily computed irreducible representation of S_n in which the entries of all the matrices are restricted to $+1$, -1, or 0. Other advantages to such a set of tableaux are listed in [**2**].

As a final note, we mention that the first footnote in [**4**, p. 53] should read "*Young IV, p. 258". Also, the "r" and "s" are interchanged in the next to last line in [**5**, p. 401]. Finally, the second and third lines in [**5**, p. 402] (Young's fourth paper

On quantitative substitutional analysis) are in error. To see this, consider the frame (3, 2). Let $X_{rs} = \delta_{r5}\delta_{s5} = Y_{rs}$, $r, s = 1, 2, \ldots, 5$. Since $M_5 = I$, $\sum X_{rs}P_r\sigma_{rs}N_sM_s = \sum Y_{rs}P_r\sigma_{rs}N_s$. Now $E - S = A_I^T$. Thus $[Y_{rs}] = [X_{rs}] \neq [X_{rs}](E - S)$. One can show that the third line in [5, p. 402] should real "$[X_{rs}] = [Y_{rs}](E - S)^T$".

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