Absolutely Convergent Fourier Series of Distributions

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Abstract. Let $S$ be a distribution (in the sense of L. Schwartz) defined on the circle $T$, and suppose that $S$ is equal to a function in $L^\infty$ on an open interval of $T$. A necessary and sufficient condition is given in order that the Fourier series of $S$ converges absolutely.

1. Introduction. The problem of characterizing the class of all Lebesgue integrable complex-valued functions on the circle $T$ (the additive group of the reals modulo $2\pi$) is a very important one in the theory of Fourier series. In [1] (see also [2]), we gave criteria for a function $f \in L^1(T)$ to have an absolutely convergent Fourier series. The criteria given in [1] are to be compared with those given by M. Riesz and S. B. Stečkin (see [2]). It seems that one of the useful aspects of the method used in [1] is that it can be extended to the case where $f$ is a distribution, and this is exactly what we intend to do in this paper. I am indebted to Paul Malliavin for helpful discussion on the subject.

2. Preliminaries and notation. With $C^\infty$ we denote the set of all $2\pi$-periodic infinitely differentiable functions.

Let $s$ be a distribution defined on $T$, which is equal to a function of the class $L^\infty$ on an open interval $I$ of $T$ containing the point $\alpha \in \mathbb{R}$. For each $\phi \in C^\infty$ set $\langle S_\alpha, \phi \rangle = \langle S, \phi_\alpha \rangle$ where $\phi_\alpha(t) = \phi(t - \alpha), \alpha \in \mathbb{R}$. Clearly $S_\alpha$ is also a distribution which is equal to a function of $L^\infty$ in an open interval containing the origin. By $S_0$ we mean the distribution $S$.

The Fourier coefficients of $S_\alpha$ are

$$\hat{S}_\alpha(n) = \langle S_\alpha, e^{-in} \rangle = \langle S, e^{-in(\alpha - \alpha)} \rangle = e^{ina} \langle S, e^{-int} \rangle = e^{ina} \hat{S}(n).$$

For $b \in \mathbb{R}$ we define, as usual, $b^+ = \max(b, 0), b^- = \max(-b, 0)$. Re $z$, Im $z$ mean the real and imaginary parts of $z$ respectively.

3. The main theorem.

Theorem. Let $S$ be a distribution (in the sense of L. Schwartz) defined on $T$. Suppose that on an open interval $I$ of $T$, $S$ is equal to a function of $L^\infty$. Then the Fourier series of $S$ is absolutely convergent if, for some $\alpha \in I$, the sequences

$$\langle (\text{Re } \hat{S}_\alpha(n))^{-\infty}_{n=-1}, \rangle \quad \text{and} \quad \langle (\text{Im } \hat{S}_\alpha(n))^{-\infty}_{n=-1}, \rangle$$

both belong to $l^1$.

The converse of the above statement also holds.
Proof. Suppose the sequences in (1) belong to \( l^1 \). We first consider the case \( \alpha = 0 \). Let \( \varphi \) be an infinitely differentiable function with support in the interval \((-\pi, \pi)\), and such that \( \hat{\varphi}(t) > 0, \hat{\varphi}(0) = 1 \), where \( \hat{\varphi} \) is the Fourier transform of \( \varphi \).

Set

\[
\varphi_{\varepsilon}(x) = \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}.
\]

For sufficiently small \( \varepsilon > 0 \), the function \( \varphi_{\varepsilon} \) is also infinitely differentiable with support in \((-\pi, \pi)\). Then, \( \varphi_{\varepsilon} \) can be extended to a \( 2\pi \)-periodic function \( \hat{\varphi}_{\varepsilon} \in C^\infty \). Put \( \varphi_{\varepsilon} \ast \hat{S} = u_{\varepsilon} \). It is known (see [3, p. 71]) that \( u_{\varepsilon} \in C^\infty \). Hence \( u_{\varepsilon} \) equals the sum of its Fourier series. We have

\[
u_{\varepsilon}(0) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(en)\hat{S}(n) = \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{\varphi}(en)\hat{S}(n).
\]

Set

\[
\sigma_{N,\varepsilon} = \sum_{n=-N}^{N} \hat{\varphi}(en)\hat{S}(n) = \sum_{n=-N}^{N} \hat{\varphi}(en)\left[ \text{Re}(\hat{S}(n)) + i \text{Im}(\hat{S}(n)) \right]
\]

\[
= \sum_{n=-N}^{N} \hat{\varphi}(en)\left[ (\text{Re} \hat{S}(n))^+ - (\text{Re} \hat{S}(n))^+ + i(\text{Im} \hat{S}(n))^+ - i(\text{Im} \hat{S}(n))^+ \right]
\]

\[
\sum_{n=-N}^{N} \hat{\varphi}(en)\text{Re} \hat{S}(n))^+ + i \sum_{n=-N}^{N} \hat{\varphi}(en)\text{Im} \hat{S}(n))^+ - i \sum_{n=-N}^{N} \hat{\varphi}(en)\text{Im} \hat{S}(n))^+.
\]

Next observe that, due to the hypothesis that \( S \) equals a function of \( L^\infty \) in \( I \), \( u_{\varepsilon}(0) \) remains uniformly bounded when \( \varepsilon \to 0 \). Furthermore since, by assumption, the sequences in (1) belong to \( l^1 \), it follows that the expressions

\[
\sum_{n=-N}^{N} \hat{\varphi}(en)(\text{Re} \hat{S}(n))^+, \quad \sum_{n=-N}^{N} \hat{\varphi}(en)(\text{Im} \hat{S}(n))^-
\]

remain uniformly bounded for all \( \varepsilon \) and \( N \). Set

\[
A_{m,n} = \hat{\varphi}\left(\frac{n}{m}\right)(\text{Re} \hat{S}(n))^+,
\]

where \( m, n \) are natural numbers.

It follows from what we have just proved that the double series \( \sum_{m,n} A_{m,n} \) is convergent since \( A_{m,n} > 0 \). We have

\[
\sum_{m,n} A_{m,n} = \lim_{N \to \infty} \left( \lim_{m \to \infty} \sum_{n=-N}^{N} \hat{\varphi}\left(\frac{n}{m}\right)(\text{Re} \hat{S}(n))^+ \right)
\]

\[
= \lim_{N \to \infty} \sum_{n=-N}^{N} (\text{Re} \hat{S}(n))^+ = \sum_{n \in \mathbb{Z}} (\text{Re} \hat{S}(n))^+ < +\infty.
\]

In a similar way we prove that

\[
\sum_{n \in \mathbb{Z}} (\text{Im} \hat{S}(n))^+ < +\infty.
\]
It follows that
\[ \sum_{n \in \mathbb{Z}} |\mathcal{S}(n)| < +\infty. \]
This proves the theorem in the case \( \alpha = 0 \).

Next assume \( \alpha \neq 0 \) and consider the distribution \( S_\alpha \). As we have noticed before, \( S_\alpha \) is a distribution which is equal to a function of \( L^\infty \) in an interval containing the origin, so that the above result holds for \( S_\alpha \). We have
\[ \sum_{n \in \mathbb{Z}} |\mathcal{S}_\alpha(n)| = \sum_{n \in \mathbb{Z}} |e^{i\alpha n} \mathcal{S}(n)| = \sum_{n \in \mathbb{Z}} |\mathcal{S}(n)| < +\infty. \]
This proves the theorem.

To prove the converse, let \( S \) be a distribution on \( T \) such that \( \sum_{n \in \mathbb{Z}} |\mathcal{S}(n)| < +\infty \). The sequence \( \langle \mathcal{S}(n) \rangle_{n \in \mathbb{Z}} \) being tempered (see [3, p. 65]) the distributions \( S_N = \sum_{|n| < N} \mathcal{S}(n)e_n \) (where \( e_n \) is the function \( x \mapsto e^{inx} \)) converge in the space of distributions, as \( N \to \infty \), to a distribution \( F \), so that \( \hat{F}(n) = \mathcal{S}(n) \ (n \in \mathbb{Z}) \). Hence \( F = S \). Now the function \( f(x) = \sum_{n \in \mathbb{Z}} \mathcal{S}(n)e^{inx} \) can be considered as a distribution. It is easily seen that, due to the uniform convergence of the last series, we have, for each \( \varphi \in C^\infty \),
\[ (S, \varphi) = \lim_{N \to \infty} (S_N, \varphi) = (1/2\pi) \int f(x)\varphi(x) \ dx = (f, \varphi), \]
which shows that \( S \) is an \( L^\infty \) function on \( T \).

One might find the above result interesting, also because of the following remark (see [1], [2]).

**Remark.** Call a numerical series \( \sum(a_n + ib_n) \ (a_n, b_n \in \mathbb{R}) \) "one sidedly absolutely convergent" (O.A.C.), iff:

(at least one of \( \sum a_n^+ \), \( \sum a_n^- \)) and (at least one of \( \sum b_n^+ \), \( \sum b_n^- \)) is finite.

Now it is possible that a series \( \sum(a_n + ib_n) \) is not O.A.C. while the series \( \sum(a_n + ib_n)e^{i\lambda} \) is O.A.C. In other words a non-O.A.C. series can, in some cases, be converted to an O.A.C. series by just multiplying each term by a factor of the form \( e^{i\lambda} \) (\( \lambda = \) some constant) or perhaps in some other way.

**Example.** Let \( c_n = a_n + ib_n \), where \( c_{2n} = 1 + i \), \( c_{2n+1} = 1 - i \), \( n = 0, 1, 2, \ldots \), and \( \lambda = \pi/4 \). Then it is easily seen that \( \sum c_n \) is not O.A.C. while \( \sum c_n e^{i\lambda} \) is.

The theorem we have just proved essentially says that, the Fourier series of a distribution converges absolutely iff \( \sum \mathcal{S}_\alpha(n) \) is O.A.C. for some \( \alpha \in \mathbb{R} \).

**REFERENCES**


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