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ABSOLUTELY CONVERGENT FOURIER SERIES OF DISTRIBUTIONS

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ABSTRACT. Let S be a distribution (in the sense of L. Schwartz) defined on the circle T, and suppose that S is equal to a function in L^∞ on an open interval of T. A necessary and sufficient condition is given in order that the Fourier series of S converges absolutely.

1. Introduction. The problem of characterizing the class of all Lebesgue integrable complex-valued functions on the circle T (the additive group of the reals modulo 2π) is a very important one in the theory of Fourier series. In **[1]** (see also **[2]**), we gave criteria for a function $f \in L^1(T)$ to have an absolutely convergent Fourier series. The criteria given in **[1]** are to be compared with those given by M. Riesz and S. B. Stečkin (see **[2]**). It seems that one of the useful aspects of the method used in **[1]** is that it can be extended to the case where f is a distribution, and this is exactly what we intend to do in this paper. I am indebted to Paul Malliavin for helpful discussion on the subject.

2. Preliminaries and notation. With C^∞ we denote the set of all 2π-periodic infinitely differentiable functions.

Let s be a distribution defined on T, which is equal to a function of the class L^∞ on an open interval I of T containing the point $\alpha \in \mathbf{R}$. For each $\varphi \in C^\infty$ set $\langle S_\alpha, \varphi\rangle = \langle S, \varphi_\alpha\rangle$ where $\varphi_\alpha(t) = \varphi(t-\alpha)$, $\alpha \in \mathbf{R}$. Clearly S_α is also a distribution which is equal to a function of L^∞ in an open interval containing the origin. By S_0 we mean the distribution S.

The Fourier coefficients of S_α are

$$\hat{S}_\alpha(n) = \langle S_\alpha, e^{-itn}\rangle = \langle S, e^{-in(t-\alpha)}\rangle = e^{in\alpha}\langle S, e^{-int}\rangle = e^{in\alpha}\hat{S}(n).$$

For $b \in \mathbf{R}$ we define, as usual, $b^+ = \max(b, 0)$, $b^- = \max(-b, 0)$. $\operatorname{Re} z$, $\operatorname{Im} z$ mean the real and imaginary parts of z respectively.

3. The main theorem.

THEOREM. *Let S be a distribution (in the sense of L. Schwartz) defined on T. Suppose that on an open interval I of T, S is equal to a function of L^∞. Then the Fourier series of S is absolutely convergent if, for some $\alpha \in I$, the sequences*

$$(1) \qquad \langle(\operatorname{Re} \hat{S}_\alpha(n))^-\rangle_{n=1}^\infty, \qquad \langle(\operatorname{Im} \hat{S}_\alpha(n))^-\rangle_{n=1}^\infty,$$

both belong to l^1.

The converse of the above statement also holds.

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PROOF. Suppose the sequences in (1) belong to l^1. We first consider the case $\alpha = 0$. Let φ be an infinitely differentiable function with support in the interval $(-\pi, \pi)$, and such that $\hat{\varphi}(t) > 0$, $\hat{\varphi}(0) = 1$, where $\hat{\varphi}$ is the Fourier transform of φ.

Set

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right), \qquad x \in \mathbf{R}.$$

For sufficiently small $\varepsilon > 0$, the function φ_ε is also infinitely differentiable with support in $(-\pi, \pi)$. Then, φ_ε can be extended to a 2π-periodic function $\tilde{\varphi}_\varepsilon \in C^\infty$. Put $\tilde{\varphi}_\varepsilon * S = u_\varepsilon$. It is known (see [3, p. 71]) that $u_\varepsilon \in C^\infty$. Hence u_ε equals the sum of its Fourier series. We have

$$u_\varepsilon(0) = \sum_n \hat{\varphi}(\varepsilon n)\hat{S}(n) = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{\varphi}(\varepsilon n)\hat{S}(n).$$

Set

$$\begin{aligned}
\sigma_{N,\varepsilon} &= \sum_{n=-N}^{N} \hat{\varphi}(\varepsilon n)\hat{S}(n) = \sum_{n=-N}^{N} \hat{\varphi}(\varepsilon n)\left[\operatorname{Re}(\hat{S}(n)) + i \operatorname{Im}(\hat{S}(n))\right] \\
&= \sum_{n=-N}^{N} \hat{\varphi}(\varepsilon n)\left[(\operatorname{Re} \hat{S}(n))^+ - (\operatorname{Re} \hat{S}(n))^- + i(\operatorname{Im} \hat{S}(n))^+ - i(\operatorname{Im} \hat{S}(n))^-\right] \\
&= \sum_{n=-N}^{N} \hat{\varphi}(\varepsilon n)(\operatorname{Re} \hat{S}(n))^+ + i \sum_{n=-N}^{N} \hat{\varphi}(\varepsilon n)(\operatorname{Im} \hat{S}(n))^+ \\
&\quad - \sum_{n=-N}^{N} \hat{\varphi}(\varepsilon n)(\operatorname{Re} \hat{S}(n))^- - i \sum_{n=-N}^{N} \hat{\varphi}(\varepsilon n)(\operatorname{Im} \hat{S}(n))^-.
\end{aligned}$$

Next observe that, due to the hypothesis that S equals a function of L^∞ in I, $u_\varepsilon(0)$ remains uniformly bounded when $\varepsilon \to 0$. Furthermore since, by assumption, the sequences in (1) belong to l^1, it follows that the expressions

$$\sum_{n=-N}^{N} \hat{\varphi}(\varepsilon n)(\operatorname{Re} \hat{S}(n))^+, \qquad \sum_{n=-N}^{N} \hat{\varphi}(\varepsilon n)(\operatorname{Im} \hat{S}(n))^-$$

remain uniformly bounded for all ε and N. Set

$$A_{m,n} = \hat{\varphi}\left(\frac{n}{m}\right)(\operatorname{Re} \hat{S}(n))^+,$$

where m, n are natural numbers.

It follows from what we have just proved that the double series $\sum_{m,n} A_{m,n}$ is convergent since $A_{m,n} \geqslant 0$. We have

$$\begin{aligned}
\sum_{m,n} A_{m,n} &= \lim_{N\to\infty}\left(\lim_{m\to\infty} \sum_{n=-N}^{N} \hat{\varphi}\left(\frac{n}{m}\right)(\operatorname{Re} \hat{S}(n))^+\right) \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} (\operatorname{Re} \hat{S}(n))^+ = \sum_{n\in\mathbf{Z}} (\operatorname{Re} \hat{S}(n))^+ < +\infty.
\end{aligned}$$

In a similar way we prove that

$$\sum_{n\in\mathbf{Z}} (\operatorname{Im} \hat{S}(n))^+ < +\infty.$$

It follows that

$$\sum_{n\in\mathbf{Z}} |\hat{S}(n)| < +\infty.$$

This proves the thoerem in the case $\alpha = 0$.

Next assume $\alpha \neq 0$ and consider the distribution S_α. As we have noticed before, S_α is a distribution which is equal to a function of L^∞ in an interval containing the origin, so that the above result holds for S_α. We have

$$\sum_{n\in\mathbf{Z}} |\hat{S}_\alpha(n)| = \sum_{n\in\mathbf{Z}} |e^{in\alpha}\hat{S}(n)| = \sum_{n\in\mathbf{Z}} |\hat{S}(n)| < +\infty.$$

This proves the theorem.

To prove the converse, let S be a distribution on T such that $\sum_{n\in\mathbf{Z}}|\hat{S}(n)| < +\infty$. The sequence $\langle\hat{S}(n)\rangle_{n\in\mathbf{Z}}$ being tempered (see [3, p. 65]) the distributions $S_N = \sum_{|n|<N} \hat{S}(n)e_n$ (where e_n is the function $x \to e^{inx}$) converge in the space of distributions, as $N \to \infty$, to a distribution F, so that $\hat{F}(n) = \hat{S}(n)$ ($n \in \mathbf{Z}$). Hence $F = S$. Now the function $f(x) = \sum_{n\in\mathbf{Z}} \hat{S}(n)e^{inx}$ can be considered as a distribution. It is easily seen that, due to the uniform convergence of the last series, we have, for each $\varphi \in C^\infty$,

$$(S, \varphi) = \lim_{N\to\infty} (S_N, \varphi) = (1/2\pi)\int f(x)\varphi(x)\,dx = (f, \varphi),$$

which shows that S is an L^∞ function on T.

One might find the above result interesting, also because of the following remark (see [1], [2]).

Remark. Call a numerical series $\Sigma(a_n + ib_n)$ ($a_n, b_n \in \mathbf{R}$) "one sidedly absolutely convergent" (O.A.C.), iff:

(at least one of $\Sigma\, a_n^+, \Sigma\, a_n^-$) and (at least one of $\Sigma\, b_n^+, \Sigma\, b_n^-$) is finite.

Now it is possible that a series $\Sigma(a_n + ib_n)$ is not O.A.C. while the series $\Sigma(a_n + ib_n)e^{i\lambda}$ is O.A.C. In other words a non-O.A.C. series can, in some cases, be converted to an O.A.C. series by just multiplying each term by a factor of the form $e^{i\lambda}$ (λ = some constant) or perhaps in some other way.

Example. Let $c_n = a_n + ib_n$, where $c_{2n} = 1 + i$, $c_{2n+1} = 1 - i$, $n = 0, 1, 2, \ldots$, and $\lambda = \pi/4$. Then it is easily seen that $\Sigma\, c_n$ is not O.A.C. while $\Sigma\, c_n e^{i\lambda}$ is.

The theorem we have just proved essentially says that, the Fourier series of a distribution converges absolutely iff $\Sigma\, \hat{S}_\alpha(n)$ is O.A.C. for some $\alpha \in \mathbf{R}$.

References

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