PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 2, October 1981

A MÜNTZ-SZASZ THEOREM FOR $C(\bar{D})$

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ABSTRACT. A Müntz-Szasz theorem is proved for the continuous functions on the closed unit disc $\bar{D}$. As a corollary it is shown that if $\gcd(n, m) = 1$, the uniformly closed algebra generated by $1, z^n, \bar{z}^m$ is $C(\bar{D})$ (Minsker).

The purpose of this paper is to derive a Müntz-Szasz type theorem for continuous functions on the closed unit disc, $\bar{D}$, in the complex plane. As an application we derive a theorem of Minsker [2]. (Also see [1].)

Let M be a subset of $\mathbf{N}^*$, the nonnegative integers. Recall that the Müntz-Szasz theorem for $[0, 1]$ says that $C[0, 1]$ is the (uniformly) closed span of $\{1, t^m: m \in M\}$ if and only if $\sum_{m \in M'} 1/m$ diverges [3], where M' denotes the nonzero m's in M. If $M' = \varnothing$, then we assume that the sum converges.

By the Stone-Weierstrass theorem (for the complex plane) the closed span of $\{z^n\bar{z}^m: n, m \in \mathbf{N}^*\}$ is $C(\bar{D})$. Let M be a subset of $\mathbf{N}^* \times \mathbf{N}^*$. Denote by $\mathfrak{M}$ the set $\{1, z^n\bar{z}^m: (n, m) \in M\}$. For any integer k, M_k will represent the set $\{m: (m, m + k) \in M\}$. We wish to characterize those subsets, M, of $\mathbf{N}^* \times \mathbf{N}^*$ so that the closed span of $\mathfrak{M}$ is $C(\bar{D})$.

THEOREM. *The closed span of $\mathfrak{M}$ is $C(\bar{D})$ if and only if*

$$\text{(1)} \qquad \textit{for every integer } j, \ \sum_{m \in M_j'} \frac{1}{m} \ \textit{diverges}.$$

PROOF. Assume that (1) holds. If $\mu \in C(\bar{D})^*$ and $\mu \perp \mathfrak{M}$, we show that μ is the zero measure. For each fixed $j \in \mathbf{N}^*$

$$\int_{\bar{D}} |z|^{2m} \bar{z}^j \, d\mu = 0$$

for all $m \in M_j$. Using (1) and the Müntz-Szasz theorem we may approximate t^{2n} uniformly on $[0, 1]$ by linear combinations of t^{2m} with $m \in M_j$ (since t^{2n} vanishes at 0). Thus $|z|^{2n}$ can be uniformly approximated on $\bar{D}$ by linear combinations of $|z|^{2m}$ so

$$\int_{\bar{D}} |z|^{2n} \bar{z}^j \, d\mu = 0, \qquad n = 1, 2, \ldots.$$

Received by the editors August 26, 1980 and, in revised form, December 5, 1980.

1980 *Mathematics Subject Classification*. Primary 30E10, 30E05.

Key words and phrases. Müntz-Szasz theorem, Stone-Weierstrass theorem, uniform approximation in the plane.

[1]The author was partially supported by a grant from the Research Grants Committee of the University of Alabama.

Similarly, if $-j \in \mathbf{N}$

$$\int_{\bar{D}} |z|^{2n} z^j \, d\mu = 0, \qquad n = 1, 2, \ldots .$$

By the Stone-Weierstrass theorem $|z|^2 \, d\mu$ is the zero measure and since μ annihilates 1, μ is the zero measure.

Conversely, suppose that (1) fails for some fixed j which we assume is nonnegative. If $M_j' = \varnothing$, take γ_0 to be any nonzero measure on [0, 1] with $\gamma_0 \perp 1$. Else $\sum_{n \in M_j'} 1/n < \infty$. Thus using the Müntz-Szasz theorem and duality there is a measure γ_0 carried by [0, 1] (which is not identically zero), with $\gamma_0 \perp 1$ and

$$\int_0^1 t^{2m+j} \, d\gamma_0 = 0 \quad \text{for } m \in M_j.$$

Define the product measure $d\gamma = d\gamma_0 \times e^{ij\theta} d\theta$, where $d\theta$ denotes Lebesgue measure on $[0, 2\pi]$. Computing with $k > 0$ in (2) and with $k > 1$ in (3):

$$\int_{\bar{D}} z^n \bar{z}^{n+k} \, d\gamma = \left(\int_0^1 t^{2n+k} \, d\gamma_0 \right) \left(\int_0^{2\pi} e^{i(j-k)\theta} \, d\theta \right), \tag{2}$$

$$\int_{\bar{D}} z^{n+k} \bar{z}^n \, d\gamma = \left(\int_0^1 t^{2n+k} \, d\gamma_0 \right) \left(\int_0^{2\pi} e^{i(j+k)\theta} \, d\theta \right). \tag{3}$$

Thus (3) vanishes for all k and (2) vanishes if $j \neq k$. When $j = k$ then by the choice of γ_0, (2) again vanishes for $n \in M_j$. Thus $\gamma \perp \mathfrak{M}$ and γ is not trivial.

COROLLARY (MINSKER [2]). *If n, m are positive integers with* $\gcd(n, m) = 1$, *then the algebra generated by* $\{1, z^n, \bar{z}^m\}$ *is* $C(\bar{D})$.

PROOF. Since $\gcd(n, m) = 1$, there exist positive integers k, l with $kn - lm = -1$. So for any positive integers j and t,

$$j(k + tm)n - j(l + tn)m = -j.$$

Thus $(z^n)^{j(k+tm)} (\bar{z}^m)^{j(l+tn)}$ is in the algebra generated by $\{1, z^n, \bar{z}^m\}$, and with our previous notation, $j(k + tm)n \in M_j$. Similar arguments hold for j nonpositive. Since $\sum_{t=1}^{\infty} (j(k + tm)n)^{-1}$ diverges, the theorem applies.

REMARKS. (a) This extension of the Müntz-Szasz theorem was based on a reduction to the one-dimensional case. We note that the version of the Stone-Weierstrass theorem which we have used (the closed span of $\{z^n \bar{z}^m : n, m \in \mathbf{N}^*\}$ on $\bar{D}$ is $C(\bar{D})$), reduces to the one-dimensional Weierstrass theorem on [0, 1] as follows:

Let $\mu \in C(\bar{D})^*$ and $\mu \perp z^j \bar{z}^k$, $j = 0, 1, \ldots,\ k = 0, 1, \ldots$. Then $\mu \perp |z|^{2n}$, $n = 0, 1, \ldots,$ so μ annihilates continuous radially symmetric functions, using the one-dimensional version. By dominated convergence μ kills all $|\mu|$-integrable radially symmetric functions. Similarly for $a \in \mathbf{C}$, μ annihilates all $|\mu|$-integrable functions radially symmetric about a. Thus the logarithmic potential $\tilde{\mu}(a) = -\int \log|z - a| \, d\mu(z)$ vanishes almost everywhere with respect to area measure in $\mathbf{C}$. Hence μ is the zero measure [4, p. 205].

(b) If K is any compact subset of $\mathbf{C}$, then the theorem gives sufficient (but not in general necessary) conditions that the closed span of $\mathfrak{M}$ is $C(K)$.

The author wishes to thank the reviewer for several helpful suggestions.

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