PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 2, October 1981

ON THE MEANS OF QUASIREGULAR AND QUASICONFORMAL MAPPINGS

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ABSTRACT. Two theorems are given regarding the means of quasiconformal and quasiregular mappings. Together they show that the principle of subordination for means of analytic functions has no analog, at least in the case of plane quasiregular mappings.

I. Introduction. Let $B^n(r) = \{x \in \mathbf{R}^n\colon |x| < r\}$, $B^n = B^n(1)$, and $f(x)$ be a quasiregular mapping of B^n. Then f is continuous, $f \in W^1_{n,\mathrm{loc}}(B^n)$, and for some constant K $(1 \leqslant K < \infty)$,

$$\|f'(x)\|^n \leqslant KJ_f(x) \quad \text{a.e.;} \tag{1.1}$$

here J_f is the Jacobian determinant, f' the Jacobian matrix, and

$$\|f'(x)\| = \sup_{|e|=1} |f'(x)\cdot e|. \tag{1.2}$$

If in addition f is injective, f is called quasiconformal.

The theory of quasiregular mappings is often referred to as a natural extension of analytic function theory, and indeed many basic properties of analytic functions extend in some form. A brief account of this along with relevant definitions can be found in the survey [5].

In the present note we shall present two results which together show that the subordination principle for means, which plays a prominent role in the theory of analytic functions [2, p. 10], has no counterpart in the theory of quasiregular mappings.

We shall use polar coordinates, denoting by $d\sigma$ the surface element on ∂B^n, and dV the volume element on $\mathbf{R}^n$.

THEOREM 1. *Let f be a quasiconformal mapping of B^n. Then there exists a $p = p(n, K)$ such that*

$$\limsup_{R\to 1} \int_{\partial B^n} |f(R\zeta)|^p \, d\sigma(\zeta) < \infty. \tag{1.3}$$

In the case $n = 2$ and f conformal, any $p < \frac{1}{2}$ suffices in (1.3) [2, p. 50]; our proof does not give this. Also if $n = 2$ and f is conformal, then the classical subordination principle says that the pth mean is bounded for any analytic g in B^2, whose range is

Received by the editors December 7, 1980.
1980 *Mathematics Subject Classification*. Primary 30C60, 30C75, 30C80.
Key words and phrases. Quasiconformal, quasiregular, means.
[1]Research of the authors supported by the National Science Foundation.

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0002-9939/81/0000-0467/$01.75

contained in $f(B^2)$. Our next example shows, at least in the case $n = 2$, that there is no corresponding effect for quasiregular mappings.

THEOREM 2. *There exists a quasiregular function g mapping B^2 to a half plane such that if $\Phi(t)$: $\mathbf{R} \to \mathbf{R}$ is any increasing function with $\Phi(t) \to \infty$ as $t \to \infty$ then*

$$\lim_{r\to 1} \int_0^{2\pi} \Phi(|g(re^{i\theta})|)\, d\theta = \infty.$$

It would be nice to have an example for arbitrary n.

II. Proof of Theorem 1. Let $f = (f_1, \ldots, f_n)$. We may assume without loss of generality that $f(0) = 0$ and thus $|f| > \varepsilon$ for some $\varepsilon > 0$, if $|x| > \frac{1}{2}$. In the course of this proof we shall use the letter C to denote constants which may not be the same at each occurrence, but which do not depend upon R and p.

Since f is absolutely continuous on almost every ray from the origin [4, p. 108] we have for any $p > 0$ and $\frac{1}{2} < R < 1$

$$\begin{aligned}
&\int_{\partial B^n} |f(R\zeta)|^p\, d\sigma(\zeta) - \int_{\partial B^n} |f(\tfrac{1}{2}\zeta)|^p\, d\sigma(\zeta) \\
&\qquad = \int_{\partial B^n} \int_{1/2}^{R} \frac{\partial |f|^p}{\partial r}\, dr\, d\sigma \\
&\qquad = p \int_{1/2}^{R} \int_{\partial B^n} |f|^{p-2}\left(f_1 \frac{\partial f_1}{\partial r} + \cdots + f_n \frac{\partial f_n}{\partial r}\right) d\sigma\, dr \\
&\qquad \leqslant p \int_{1/2}^{R} \int_{\partial B^n} |f|^{p-1}\left(\left(\frac{\partial f_1}{\partial r}\right)^2 + \cdots + \left(\frac{\partial f_n}{\partial r}\right)^2\right)^{1/2} d\sigma\, dr.
\end{aligned}$$

Applying (1.1), (1.2) and then Hölder's inequality we obtain for $\frac{1}{2} < R < 1$

$$\begin{aligned}
\int_{\partial B^n} |f(R\zeta)|^p\, d\sigma(\zeta) &\leqslant K^{1/n} p \int_{1/2}^{R} \int_{\partial B^n} |f|^{p-1} J_f^{1/n}\, d\sigma\, dr + C \\
&\leqslant K^{1/n} p \left(\int_{1/2}^{R} \int_{\partial B^n} |f|^{-n(p+1)} J_f r^{n-1}\, d\sigma\, dr\right)^{1/n} \\
&\quad \cdot \left(\int_{1/2}^{R} \int_{\partial B^n} |f|^{2pn/(n-1)} r^{-1}\, d\sigma\, dr\right)^{(n-1)/n} + C.
\end{aligned} \tag{2.1}$$

Now,

$$\begin{aligned}
\int_{1/2}^{R} \int_{\partial B^n} |f|^{-n(p+1)} J_f r^{n-1}\, d\sigma\, dr &= \int_{B^n(R) - B^n(1/2)} |f|^{-n(p+1)} J_f\, dV \\
&\leqslant \int_{\mathbf{R}^n - B^n(\varepsilon)} |x|^{-n(p+1)}\, dV(x) < \infty.
\end{aligned} \tag{2.2}$$

Now, $f(B^n) \neq R^n$ [4, p. 53] so if τ is an omitted value we have [3, p. 73] $|f(x)| \leqslant |\tau| + C|\tau|((1 + r)/(1 - r))^\alpha$, $\alpha = \alpha(K, n)$ and thus

$$\int_{1/2}^{R} \int_{\partial B^n} |f|^{2pn/(n-1)} r^{-1}\, d\sigma\, dr \leqslant C \int_{1/2}^{R} (1 - r)^{-2pn\alpha/(n-1)}\, dr + C. \tag{2.3}$$

Since the right-hand side of (2.3) is finite for sufficiently small $p > 0$, the result follows from (2.1)–(2.3).

III. Proof of Theorem 2. For a Borel subset E of $[0, 2\pi)$, let χ_E be its characteristic function, and $|E| = \int \chi_E \, d\theta$.

From [1, p. 139] we obtain a quasiconformal mapping ϕ of B^2 onto itself such that on ∂B^2 the measure μ defined by $\mu(E) = |\phi(E)|$ is singular with respect to $d\theta$. Let $E \subseteq \partial B^2$ such that $|E| = 0$, $\mu(E) > 0$, and let $V_1 \supseteq V_2 \supseteq \dots$ be open subsets of $[0, \pi)$ such that $|V_n| = 2^{-n}$ and $\bigcap_n V_n \supseteq E$. Define $f(\theta) = \Sigma \chi_{V_n}(\theta)$. Then, $f \in L^2(d\theta)$ since

$$\int_0^{2\pi} |f|^2 \, d\theta \leqslant \sum_{n=1}^{\infty} \int_{V_n - \bigcup_{j>n} V_j} \left(\sum_{k=1}^{\infty} \chi_{V_k} \right)^2 d\theta$$
$$\leqslant \sum_{n=1}^{\infty} \int_{V_n} n \left(\sum_{k=1}^{n} \chi_{V_k} \right) d\theta \leqslant \sum_{n=1}^{\infty} n^2 2^{-n} < \infty.$$

Thus, we may take the Poisson integral of f in B^2, which we continue to denote by f. Let $\tilde{f}$ be a conjugate of f, and $F = f + i\tilde{f}$.

If $g = F \circ \phi^{-1}$ then, g is quasiregular, and $\mathfrak{Re}\, g > 0$ in B^2, and

$$\begin{aligned} \int_0^{2\pi} \Phi(|g(re^{i\theta})|) \, d\theta &\geqslant \int_0^{2\pi} \Phi(f \circ \phi^{-1}(re^{i\theta})) \, d\theta \\ &\geqslant \int_{\phi(V_n)} \Phi(f \circ \phi^{-1}(re^{i\theta})) \, d\theta \quad \forall\, n = 1, 2, \dots . \end{aligned} \tag{3.1}$$

Now, the right-hand side of (3.1) is bounded below by $\int_{\phi(V_n)} \Phi(h_n \circ \phi^{-1}(re^{i\theta})) \, d\theta$ where h_n is the Poisson integral of $n\chi_{V_n}$. But $h_n(re^{i\theta})$ is continuous for $0 < r \leqslant 1$, $\theta \in V_n$ since V_n is open, and ϕ^{-1} is continuous on $\bar{B}^2$. Thus, the limit of $\int_{\phi(V_n)} \Phi(h_n \circ \phi^{-1}(re^{i\theta})) \, d\theta$ exists as $r \to 1$. Since

$$\int_{\phi(V_n)} \Phi(n) \, d\theta = \Phi(n)\mu(V_n) \geqslant \Phi(n)\mu(E), \qquad 0 < \mu(E),$$

and $\Phi(n)$ can be made arbitrarily large, it follows that

$$\int_0^{2\pi} \Phi(|g(re^{i\theta})|) \, d\theta \to \infty \quad \text{as } r \to 1.$$

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