PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 2, October 1981

INJECTIVE MATRIX FUNCTIONS[1]

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ABSTRACT. Univalence of holomorphic (scalar) functions $f(z)$ is generalized to injectivity of holomorphic matrix functions $V(z) = (v_{ik}(z))_1^n$. Local injectivity is characterized by $|V'(z_0)| \neq 0$ ($|A| = \det A$). The classes S and Σ are defined as in the scalar case. For each class a sufficient condition is proved and a necessary condition is conjectured.

1. Introduction. Injective vector and matrix functions are defined as follows [4]. Let the (column) vector $v(z) = (v_1(z), \ldots, v_n(z))^T$ be holomorphic in a domain D of the z-plane (i.e. each component $v_k(z)$ is holomorphic in D). *$v(z)$ is called injective in D, if*

$$v(z_1) \neq v(z_2), \qquad z_1, z_2 \in D,\ z_1 \neq z_2. \tag{1}$$

An $n \times n$ matrix $V(z) = (v_{ik}(z))_1^n$, holomorphic in D, is called injective there, if for every constant vector c, $c \neq 0$, the holomorphic vector function

$$v(z) = V(z)c \tag{2}$$

is injective in D. The following criterion for matrix injectivity (cf. [4]) will be used throughout this paper. (The determinant of the matrix A is denoted by $|A|$.)

LEMMA 1. *The holomorphic matrix $V(z) = (v_{ik}(z))_1^n$ is injective in D if and only if*

$$|V(z_1) - V(z_2)| \neq 0, \qquad z_1, z_2 \in D,\ z_1 \neq z_2. \tag{3}$$

PROOF. (3) holds if and only if $(V(z_1) - V(z_2))c = 0$ implies $c = 0$; hence (3) is equivalent to (1), for all $v(z)$ given by (2), $c \neq 0$.

In the scalar case, $n = 1$, $|f(z_1) - f(z_2)| \neq 0$, $z_1, z_2 \in D$, $z_1 \neq z_2$, is the definition of univalence of the holomorphic function $f(z)$ in D. Matrix injectivity is thus a generalization of univalence. Sufficient conditions for injectivity of $V(z)$, given by bounds for norms of $V''(z)V'(z)^{-1}$ and by bounds for norms of the Schwarzian derivative

$$\{V(z), z\} = \left(V''(z)V'(z)^{-1}\right)' - \tfrac{1}{2}\left(V''(z)V'(z)^{-1}\right)^2, \tag{4}$$

were found by the use of matrix differential equations [5, Theorems 3.3–3.6, 4.2′ and 4.3′]. This paper contains some results (and a conjecture) about injectivity, which are not obtained by differential equations.

Received by the editors October 1, 1980.
1980 *Mathematics Subject Classification.* Primary 30C55; Secondary 15A54.
Key words and phrases. Univalent functions, injective matrix functions.
[1]This research was sponsored by the fund for the Encouragement of Research at the Technion.

0002-9939/81/0000-0472/$02.50

2. Local injectivity and decidable classes.

THEOREM 1. *Let the matrix* $V(z) = (v_{ik}(z))_1^n$ *be holomorphic in a domain* D *and let* $V'(z) = (v'_{ik}(z))$ *be its derivative. If*

$$|V'(z_0)| \neq 0, \tag{5}$$

$z_0 \in D$, *then* $V(z)$ *is locally injective at* z_0. *Conversely, the condition* (5) *is also necessary for local injectivity at* z_0, *except if* $|V'(z)| \equiv 0$ *in* D.

PROOF. Let z_1 and z_2 be points near z_0 in D. Using

$$v_{ik}(z_j) = v_{ik}(z_0) + (z_j - z_0)v'_{ik}(z_0) + \dots, \qquad j = 1, 2, i, k = 1, \dots, n,$$

we obtain

$$V(z_1) - V(z_2) = (z_1 - z_2)V'(z_0) + o(|z_1 - z_0| + |z_2 - z_0|).$$

(5) thus implies (3) for z_1 and z_2 near z_0, $z_1 \neq z_2$.

To prove the necessity[2] of (5), we assume, by negation $|V'(z_0)| = 0$ and $|V'(z)| \not\equiv 0$. Let the disk $|z - z_0| \leq r$ belong to D, and choose ε, $0 < \varepsilon < r$. Let $\{h_k\}_1^\infty$ be a sequence of complex numbers such that $0 < |h_k| < \varepsilon/2$ and $\lim_{k\to\infty} h_k = 0$. We define the scalar functions

$$f_k(z) = |h_k^{-1}(V(z + h_k) - V(z))|, \qquad |z - z_0| < \varepsilon/2,$$

$k = 1, 2, \dots$. Then $\lim_{k\to\infty} f_k(z) = |V'(z)| = f_0(z)$, $|z - z_0| < \varepsilon/2$, and it is easily seen that the convergence is uniform in $|z - z_0| < \varepsilon/2$. As $f_0(z_0) = 0$, $f_0(z) \not\equiv 0$, it follows by Hurwitz's theorem that, for large k, $f_k(z)$ has at least one zero z_k in the above disk; i.e. $|V(z_k + h_k) - V(z_k)| = 0$, and $V(z)$ is thus not injective in $|z - z_0| \leq \varepsilon$. As ε, $0 < \varepsilon < r$, was arbitrary, this completes the proof.

The following example shows that, if $|V'(z)| \equiv 0$, $V(z)$ may be injective in the whole plane:

$$V(z) = \begin{pmatrix} 2z^3 & 3z^2 \\ 3z^2 & 6z \end{pmatrix}; \qquad |V'(z)| \equiv 0,$$

and

$$|V(z_1) - V(z_2)| = 3(z_1 - z_2)^4.$$

There are some classes of matrix functions $V(z)$ for which it is easy to decide whether $V(z)$ is injective or not. (The eigenvalues of the $n \times n$ matrix A are denoted $\lambda_j = \lambda_j(A)$, $j = 1, \dots, n$.)

THEOREM 2. (a) *A triangular matrix* $V(z) = (v_{ik}(z))_1^n$ *is injective in a domain* D *if and only if each diagonal element* $v_{kk}(z)$, $k = 1, \dots, n$, *is univalent in* D.

(b) *Let* A *be a constant matrix. The function* $V(z) = \exp(zA)$ *is injective in a domain* D *if and only if* $|A| = \prod_{j=1}^n \lambda_j \neq 0$ *and* $2m\pi i/\lambda_j \notin D - D$, $j = 1, \dots, n$, $m = 1, 2, \dots$. *In particular,* $V(z)$ *is injective in* $|z| < 1$ *if and only if* $0 < |\lambda_j| \leq \pi$, $j = 1, \dots, n$ [4].

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[2]I am thankful to Professor D. London for this proof.

(c) *The binomial* $P_m(z) = zI + z^m A_m$ *is injective in* $|z| < 1$ *if and only if* $|\lambda_j(A_m)| \leqslant 1/m$, $j = 1, \ldots, n$. *Similarly* $Q_m(z) = zI + (1/z^m)B_m$ *is injective in* $|z| > 1$ *if and only if* $|\lambda_j(B_m)| \leqslant 1/m, j = 1, \ldots, n$.

PROOF. (a) This follows directly from Lemma 1: $V(z_1) - V(z_2)$ is also triangular; hence its eigenvalues are the diagonal elements $v_{kk}(z_1) - v_{kk}(z_2)$, $k = 1, \ldots, n$.

(b) $|V(z_1) - V(z_2)| = |\exp(z_1 A) - \exp(z_2 A)| = |\exp(z_1 A)|\,|I - \exp(z_2 - z_1)A|$.

Hence $|V(z_1) - V(z_2)| = 0$ if and only if there exists an eigenvalue such that $\lambda_j(\exp(z_2 - z_1)A) = 1$. As $\lambda_j(\exp(z_2 - z_1)A) = \exp((z_2 - z_1)\lambda_j(A))$ this happens only if either $\lambda_j(A) = 0$ or $(z_2 - z_1)\lambda_j(A) = \pm 2m\pi i$, $m = 1, 2, \ldots$.

(c) $|P_m(z_1) - P_m(z_2)| = (z_1 - z_2)^m |I + (z_1^{m-1} + z_1^{m-2} z_2 + \cdots + z_2^{m-1})A_m|$.

For $|z_1| < 1$, $|z_2| < 1$, $z_1 \neq z_2$, the function $w(z_1, z_2) = z_1^{m-1} + z_1^{m-2} z_2 + \ldots + z_2^{m-1}$ takes all values w in the disk $|w| < m$, and only those. Hence if $|\lambda_j(A_m)| > 1/m$, for a given j, then we may find z_1 and z_2, $z_1 \neq z_2$, in $|z| < 1$ such that $w(z_1, z_2) = -1/\lambda_j(A_m)$ and then $|P_m(z_1) - P_m(z_2)| = 0$. If $|\lambda_j(A_m)| \leqslant 1/m$ for all j, then all eigenvalues of $w(z_1, z_2)A_m$ lie in $|\lambda| < 1$ and hence $|P_m(z_1) - P_m(z_2)| \neq 0$. A similar proof applies for the second half of (c).

3. The classes S and Σ. Injectivity is preserved under conformal mappings. If $z = \phi(\zeta)$ maps Δ onto D and if $V(z)$ is injective in D, then $V(\phi(\zeta)) = \tilde{V}(\zeta)$ is injective in Δ. For the study of injectivity in simply connected domains we may thus restrict ourselves to the unit disk. Injectivity is also preserved under linear mappings of V.

LEMMA 2. *Let A, B and C be constant $n \times n$ matrices, $|A| \neq 0$, $|B| \neq 0$. The holomorphic matrix $V(z) = (v_{ik}(z))_1^n$ is injective in D if and only if the same holds for the matrix $\tilde{V}(z) = AV(z)B + C$.*

This follows immediately from Lemma 1.

Let $\{C_m\}_0^\infty$ be a sequence of (constant) $n \times n$ matrices and let $V(z) = C_0 + zC_1 + z^2 C_2 + \ldots$ be holomorphic in $|z| < 1$. We may assume local injectivity; hence, if we disregard the exceptional case $|V'(z)| \equiv 0$, it follows that $|V'(0)| = |C_1| \neq 0$. Instead of the above function it suffices to consider the function $(V(z) - C_0)C_1^{-1}$; i.e. we may assume that $V(0) = 0$ and $V'(0) = I$. In analogy to the scalar case we say that *the function*

$$V(z) = zI + z^2 A_2 + \ldots \tag{6}$$

belongs to the class S if $V(z)$ is holomorphic and injective in $|z| < 1$. The following generalization of a well-known scalar result [6, p. 212], [3, p. 44] holds.

THEOREM 3. *Let $\{A_m\}_2^\infty$ be a sequence of $n \times n$ matrices and let $\|A\|$ be a given matrix norm. If*

$$\sum_{m=2}^{\infty} m\|A_m\| \leqslant 1, \tag{7}$$

then the function $V(z)$, given by (6), belongs to S.

PROOF. (7) implies $\overline{\lim}\|A_m\|^{1/m} \leqslant 1$; hence r, the radius of convergence of the power series (6) satisfies $r = (\overline{\lim}\|A_m\|^{1/m})^{-1} \geqslant 1$ and the function $V(z)$ is holomorphic in $|z| < 1$. (Note that $r = \min r_{ik}$, where r_{ik} is the radius of the scalar power series $\delta_{ik}z + \sum_{m=2}^{\infty} a_{ik}^{(m)}z^m$, $i, k = 1, \ldots, n$; $A_m = (a_{ik}^{(m)})_1^n$, $m = 2, \ldots$. This follows easily from the existence of positive constants c_1 and c_2 such that the inequality

$$c_1 \max|a_{ik}| \leqslant \|A\| \leqslant c_2 \max|a_{ik}|$$

holds for the given norm and any matrix $A = (a_{ik})_1^n$ [1].)

To prove injectivity, let z_1 and z_2 be two distinct points in $|z| < 1$. By (6),

$$V(z_1) - V(z_2) = (z_1 - z_2)(I + A(z_1, z_2))$$

where $A(z_1, z_2) = (z_1 + z_2)A_2 + (z_1^2 + z_1z_2 + z_2^2)A_3 + \ldots$. For all eigenvalues $\lambda_j = \lambda_j(A(z_1, z_2)), j = 1, \ldots, n$, it follows by (7) that

$$|\lambda_j| \leqslant \|A(z_1, z_2)\| \leqslant \sum_2^{\infty} m\|A_m\| \leqslant 1.$$

Indeed, we have strict inequality $\|A(z_1, z_2)\| < \sum_2^{\infty} m\|A_m\|$, except in the case where all $A_m = 0$. Hence, always $|\lambda_j| < 1$ and all the eigenvalues of $I + A(z_1, z_2)$ are different from zero. As

$$|V(z_1) - V(z_2)| = (z_1 - z_2)^n|I + A(z_1, z_2)|,$$

this implies $|V(z_1) - V(z_2)| \neq 0$.

Theorem 3 is sharp. If the matrix norm satisfies $\|I\| = 1$, as happens for all induced norms, then no summand $m\|A_m\|$ of assumption (7) can be replaced by $(m - \varepsilon)\|A_m\|$, $0 < \varepsilon < m$. To show this we note that the function

$$f_m(z) = z - (m - \varepsilon)^{-1}z_m$$

is not univalent in $|z| < 1$, $m = 2, \ldots$. Hence $V_m = zI - (m - \varepsilon)^{-1}z^mI$ is not injective in $|z| < 1$ and in this case $(m - \varepsilon)\|A_m\| = 1$.

Let Σ be the class of functions

$$W(z) = zI + B_0 + z^{-1}B_1 + \ldots \tag{8}$$

that are holomorphic and injective in $|z| > 1$. The following analogue of Theorem 3 holds.

THEOREM 4. *Let $\{B_m\}_0^{\infty}$ be a sequence of $n \times n$ matrices and let $\|B\|$ be a given matrix norm. If*

$$\sum_{m=1}^{\infty} m\|B_m\| \leqslant 1, \tag{9}$$

then the function $W(z)$, given by (8), *belongs to Σ.*

The proof is similar to the proof of Theorem 3, and the result is again sharp.

Well-known necessary conditions for scalar functions $f(z) = z + a_2z^2 + \ldots$ and $g(z) = z + b_0 + b_1z^{-1} + \ldots$ to belong to the classes S and Σ suggest similar necessary conditions in the matrix case. Let $\rho(A)$ denote the spectral radius of the

$n \times n$ matrix A; i.e. $\rho(A) = \max|\lambda_j(A)|$. *We conjecture that* (a) *if* $V(z) = zI + z^2A_2 + \dots$ *belongs to* S, *then*

$$\rho(A_2) \leqslant 2; \tag{10}$$

and (b) *if* $W(z) = zI + B_0 + z^{-1}B_1 + \dots$ *belongs to* Σ, *then*

$$\rho(B_1) \leqslant 1. \tag{11}$$

The following arguments support these conjectures.

(i) The spectral radius $\rho(A_2)$ ($\rho(B_1)$) seems to be the appropriate generalization of the absolute value $|a_2|$ ($|b_1|$). Indeed the classes S and Σ are invariant under similarities induced by constant matrices T, $|T| \neq 0$; if $V(z) \in S$, then also $\tilde{V}(z) = TV(z)T^{-1} \in S$. Such a similarity acts separately on each coefficient; i.e. if $V(z)$ is given by (6) then $\tilde{V}(z) = zI + z^2\tilde{A}_2 + \dots$, where $\tilde{A}_2 = TA_2T^{-1}$. Hence, $\rho(\tilde{A}_2) = \rho(A_2)$. As norms are not bounded from above under similarities, no necessary condition can be of the form $\|A_2\| < c_2$ ($\|B_1\| < c_1$). E.g., set

$$V(z) = zI + z^2\begin{pmatrix} 1/2 & 1 \\ 0 & 1/2 \end{pmatrix}.$$

By Theorem 2(c), $V(z) \in S$. Let $T = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, $a \neq 0$, $b \neq 0$; then

$$\tilde{A}_2 = TA_2T^{-1} = \begin{pmatrix} 1/2 & a/b \\ 0 & 1/2 \end{pmatrix}$$

and $\|\tilde{A}_2\| \to \infty$ as $(a/b) \to \infty$.

(ii) The inequalities (10) and (11) hold for all triangular matrices $V(z) \in S$ and $W(z) \in \Sigma$, and are sharp in these cases. This follows, by Theorem 2(a), from the necessity and sharpness of the scalar conditions $|a_2| \leqslant 2$ and $|b_1| \leqslant 1$.

(iii) $V(z) \in S$ is similar to a triangular matrix function $\tilde{V}(z) \in S$, $\tilde{V}(z) = TV(z)T^{-1}$, if and only if all the matrices $\tilde{A}_m = TA_mT^{-1}$ are triangular. If there exist scalar polynomials $p_m(\lambda)$ and a constant matrix A, such that $A_m = p_m(A)$, $m = 2, \dots,$ then the matrices A_m can be simultaneously transformed into triangular form (say by using T which transforms A into its Jordan form). Hence, for $V(z) \in S$, with $A_m = p_m(A)$, $m = 2, \dots,$ the inequality (10) holds. Similarly, (11) is necessary for an analogous subclass of Σ.

We remark that conjecture (a) is equivalent to the following conjecture: if $V(z)$ is holomorphic and injective in $|z| < 1$, and $|V'(z)| \not\equiv 0$, then

$$\rho\left(V''(z)V'(z)^{-1}\right) \leqslant \frac{1 + 2|z|}{1 - |z|^2}, \qquad |z| < 1, \tag{12}$$

(cf. [3, pp. 20–21]). If the inequality (11) is valid for all $W(z) \in \Sigma$, then the inequality

$$\rho(\{V(z), z\}) \leqslant \frac{6}{(1 - |z|^2)^2}, \qquad |z| < 1, \tag{13}$$

holds for all functions $V(z)$ which are holomorphic and injective in $|z| < 1$, $|V'(z)| \not\equiv 0$ (cf. [2]). We note also that (13) is satisfied in all cases in which the sufficient conditions of Theorems 4.2′ and 4.3′ of [5] hold.

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