

A FORMULA BY V. M. ADAMJAN, D. Z. AROV AND M. G. KREIN

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ABSTRACT. For F in L^∞ of the unit circle, Adamjan, Arov and Krein obtained a formula describing all functions in the coset $F + H^\infty$ of norms not exceeding 1. A proof of this result using classical ideas of Nevanlinna and Schur is given.

Let F be a bounded measurable function on the unit circle $T = \{z: |z| = 1\}$, and form the set

$$B_F = \{F - h: h \in H^\infty, \|F - h\|_\infty < 1\},$$

where H^∞ denotes the space of bounded measurable functions on T which extend to be analytic in $D = \{z: |z| < 1\}$ by means of the Poisson integral formula. The norm " $\|\cdot\|_\infty$ " denotes the usual essential supremum on T .

In a series of papers [1], [2], [3], V. M. Adamjan, D. Z. Arov and M. G. Krein have obtained very detailed information about the set B_F . Their proofs use the theory of bounded operators on Hilbert space and basic knowledge about the classical Hardy spaces.

The object of this paper is to give an elementary proof of one of their results:

THEOREM (ADAMJAN, AROV AND KREIN). *Suppose B_F contains at least two functions. Then there are analytic functions p and q in D such that*

$$B_F = \left\{ \frac{\bar{q} - \bar{p}w}{p - qw} : w \in H^\infty, \|w\|_\infty < 1 \right\}.$$

REMARK. The functions p and q have boundary values defined almost everywhere on T , and in the above description of B_F , all functions should be considered on the unit circle T only. Two special cases of the above problem should be mentioned:

- (i) I. Schur's coefficient problem [9], [10],
- (ii) R. Nevanlinna's work in [7], [8] on interpolation by bounded analytic functions.

What we do here is to prove the above theorem by combining ideas of Schur and Nevanlinna. In this way we obtain a proof completely based on classical complex analysis.

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PROOF OF THE THEOREM. We let σ_N denote the N th Fejer kernel and form the trigonometric polynomials $F_N = F * \sigma_N$, $N = 1, 2, \dots$. As B_F corresponds to F , we define $B_N = \{F_N - h: h \in H^\infty, \|F_N - h\|_\infty < 1\}$. If N is sufficiently large, B_N has at least two members. The following formula is due to I. Schur [9], [10] although it is not explicitly stated there:

$$(1) \quad B_N = \frac{\bar{q}_N - \bar{p}_N w}{p_N - q_N w}, \quad w \in H^\infty, \|w\|_\infty < 1,$$

where p_N and q_N are polynomials constructed by the Schur algorithm [10] in terms of the Fourier coefficients of F_N .

We list some properties of p_N and q_N which are part of Schur's construction:

$$(2) \quad p_N(z) \neq 0, \quad |z| < 1,$$

$$(3) \quad q_N(0) = 0.$$

By a suitable normalization, we may also assume

$$(4) \quad |p_N|^2 - |q_N|^2 \equiv 1 \quad \text{on } T.$$

For details, see [10, pp. 138–139], for example. By (4), equality (1) also can be written as

$$(5) \quad B_N = \left\{ \frac{\bar{q}_N}{p_N} - \frac{w}{p_N(p_N - q_N w)} : w \in H^\infty, \|w\|_\infty < 1 \right\}.$$

We now fix $z \in D$. We define \bar{q}_N/p_N at z by means of the Poisson integral, and then study the linear transformation

$$T_z: w \rightarrow \frac{\bar{q}_N}{p_N}(z) - \frac{w}{p_N(z)(p_N(z) - q_N(z))},$$

which necessarily takes D into itself. This is one of the central ideas in Nevanlinna's paper [8].

The radius of the disc $T_z(D)$ is easily computed to be

$$\rho_N(z) = (|p_N(z)|^2 - |q_N(z)|^2)^{-1}.$$

Since B_F has more than one member, we can find $z \in D$, $\delta > 0$ and an integer N_0 such that

$$\rho_N(z) \geq \delta, \quad N \geq N_0.$$

By (2)–(4) and Schwartz's lemma, we can thus conclude that the sequence $\{|p_N(z)|\}_N$ is bounded. Applying the Harnack principle to the positive harmonic function $\log|p_N|$ in D , we get that $\{p_N\}$ is uniformly bounded on compact subsets of D . We can thus find subsequences $\{p_{N_k}\}$ and $\{q_{N_k}\}$ which are uniformly convergent on compact subsets of D to functions p and q respectively. We may also assume that there is a function $r \in L^\infty$ such that $\bar{q}_{N_k}/p_{N_k} \rightarrow r$ in the w^* -topology.

Since $|q(z)| < |p(z)|$, $z \in D$, and $p^{-1} \in H^\infty$, both p and q have nontangential limits almost everywhere on T by Fatou's theorem [4, p. 13].

From the representation formula (5), it follows by an easy normal family argument that

$$(6) \quad B_F = \left\{ r - \frac{w}{p(p - qw)}, w \in H^\infty, \|w\|_\infty \leq 1 \right\}.$$

If we knew that $r = \bar{p}/q$ and $|p|^2 - |q|^2 = 1$ held almost everywhere on T , the formula for B_F given by Adamjan, Arov and Krein would easily follow from (6).

To complete our proof, we shall therefore show that $\{p_{N_k}\}$ and $\{q_{N_k}\}$ converge in measure on T .

To obtain such a conclusion, we claim

$$(*) \quad \text{If } w = w_\alpha = e^{i\alpha}, \alpha \text{ real, then } |r - w_\alpha/p(p - qw_\alpha)| = 1 \text{ almost everywhere on } T.$$

Assume the claim is proved, and put $I_\alpha = r - w_\alpha/p(p - qw_\alpha)$. Define also

$$I_{\alpha,k} = \frac{\bar{q}_{N_k}}{p_{N_k}} - \frac{w_\alpha}{p_{N_k}(p_{N_k} - q_{N_k}w_\alpha)}, \quad k = 1, 2, \dots$$

Following Heins' argument [6, p. 256], we see that $I_{\alpha,k} \rightarrow I_\alpha$ in $L^2(d\theta)$ as $k \rightarrow \infty$.

If we pick three distinct values of α in $[0, 2\pi)$, we easily deduce that

$$p_{N_k} \rightarrow p \quad \text{and} \quad q_{N_k} \rightarrow q,$$

where the convergence is in measure on T .

It remains to prove the claim (*). This claim is of course an immediate consequence of the formula we are trying to establish. Recently J. Garnett [5] has given a proof using methods from functional analysis. A result like (*) was apparently first proved by R. Nevanlinna in connection with the Pick-Nevanlinna interpolation problem [8, Satz 7].

Since we want to give a proof along the lines of classical complex analysis, we follow Nevanlinna's idea in [8, pp. 49-51]. So we assume $|I_\alpha| < 1 - 2r < 1$ on a set E of positive linear measure, and shall try to obtain a contradiction.

We put $I_k = I_\alpha * \sigma_{N_k}$, $k = 1, 2, \dots$, and write

$$(7) \quad I_k = \frac{\bar{q}_{N_k} - \bar{p}_{N_k}w_k}{p_{N_k} - q_{N_k}w_k} = \frac{\bar{q}_{N_k}}{p_{N_k}} - \frac{w_k}{p_{N_k}(p_{N_k} - q_{N_k}w_k)}$$

where $w_k \in H^\infty$, $\|w_k\|_\infty \leq 1$, $k = 1, 2, \dots$.

Since $I_k \rightarrow I$ almost everywhere on T , there is a subset $E_1 \subset E$ of positive measure, such that $|I_k| < 1 - r$ on E_1 if $k \geq K_0$. Following Nevanlinna on pp. 49-50 in [8], we can also deduce that

$$|q_{N_k}/p_{N_k}| \leq t < 1$$

on a subset E_2 of E_1 having positive measure. If $[z_1, z_2]$ denotes the noneuclidean distance between z_1 and z_2 in D , we have from (7) that

$$(8) \quad [w_k(e^{i\theta}), 0] = \left[I_k, \frac{\bar{q}_{N_k}}{p_{N_k}} \right] < [I_k, 0] + \left[\frac{\bar{q}_{N_k}}{p_{N_k}}, 0 \right] < 1 - r + t$$

for k sufficiently large and $e^{i\theta} \in E_2$. But this is a contradiction. Using (7) and the above expression for I_α , it is easy to see that $w_k(0) \rightarrow w_\alpha$ as $k \rightarrow \infty$. Since

$\|w_k\|_\infty < 1$, we must have that $w_k \rightarrow w_\alpha$ uniformly on compact subsets of D . Since w_α is a constant of modulus one, we can clearly not have $|w_k| < 1 - t_0 < 1$ on E_2 if k is sufficiently large; but this is exactly what (8) tells us. The proof is complete.

ADDED IN PROOF (January 22, 1981). The author has recently learned of a proof due to John Garnett of the result of Adamjan, Arov and Krein which is different both from the original proof and from the one above. It can be found in Chapter IV, §5 of Garnett's forthcoming book, *Bounded analytic functions* (Academic Press).

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