A CHARACTERIZATION OF COMPLETE METRIC SPACES

FRANCIS SULLIVAN

Abstract. A general formulation of the completeness argument used in the Bishop-Phelps Theorem and many other places has been given by Ekeland. It is shown that Ekeland’s formulation characterizes complete metric spaces.

A central idea in the proof of the Bishop-Phelps Theorem is the use of norm completeness and a partial ordering to produce a point where a linear functional attains its supremum on a closed bounded convex set. In fact, this completeness technique is useful in many situations as has been described in the surveys of Phelps [7] and Brezis-Browder [1]. Recently Ekeland [3] has given a very general formulation of this technique and has applied it to a wide variety of problems [4]. In the present note we show that Ekeland’s formulation is actually equivalent to completeness for metric spaces.

Ekeland’s Theorem may be stated as follows:

Theorem 1. Let \((M, d)\) be a complete metric space, and \(F: M \to \mathbb{R} \cup \{+\infty\}\) a lower semicontinuous function, \(F \neq +\infty\), bounded from below. Let \(\epsilon > 0\) be given and a point \(u \in M\) such that

\[
F(u) < \inf_{M} F + \epsilon.
\]

Then there exists a point \(v \in M\) such that

(i) \(F(v) < F(u)\).

(ii) \(d(u, v) < 1\).

(iii) For all \(w \neq v\), \(F(w) + \epsilon d(v, w) > F(v)\).

This result characterizes completeness of \(M\) in the following sense.

Theorem 2. Let \((M, d)\) be a metric space. Then \(M\) is complete if and only if for every continuous \(F: M \to \mathbb{R} \cup \{+\infty\}\), \(F \neq +\infty\), bounded from below and every \(\epsilon > 0\) there is a point \(v \in M\) satisfying

(i) \(F(v) < \inf_{M} F + \epsilon\) and (iii) above.

Proof. The “only if” direction is immediate from Theorem 1. For the converse assume that \((M, d)\) is an arbitrary metric space satisfying the hypotheses. Let \((y_n) \subseteq M\) be a Cauchy sequence and consider the function \(F: M \to \mathbb{R}\) given by

\[
F(x) = \lim_{n} d(y_n, x).
\]
The function $F$ is continuous and, since $(y_n)$ is Cauchy, $\inf F = 0$. To show completeness we produce a $v \in M$ such that $F(v) = 0$.

Choose any $0 < \varepsilon < 1$. Now, from (i) and (iii), there is a $v \in M$ with $F(v) < \varepsilon$ and $F(w) + \varepsilon d(w, v) > F(v)$ for all $w \neq v$. From the definition of $F$ and the fact that $(y_n)$ is Cauchy we can take $w = y_p$ for $p$ large such that $F(w)$ is arbitrarily small and $d(w, v) < \varepsilon + \eta$ for any $\eta > 0$, because $F(v) < \varepsilon$. Using condition (iii) we get that, in fact, $F(v) < \varepsilon^2$. Repeating the argument we may conclude that $F(v) < \varepsilon^n$ for all $n > 1$ and so $F(v) = 0$ as required. Q.E.D.


References


Department of Mathematics, Catholic University of America, Washington, D.C. 20064

Mathematisch Instituut, Katholieke Universiteit, Toernooiveld, 6525 ED Nijmegen, The Netherlands