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ON POWER-BOUNDED OPERATORS AND THE POINTWISE ERGODIC PROPERTY[1]

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ABSTRACT. We construct a power-bounded isomorphism T: $L_p \to L_p$, for $1 < p < 2$, such that $[n^{-1}\Sigma_{i=0}^{n-1} T^i f](t)$ is not a.e. convergent. T^{-1} is also power-bounded.

1. Introduction, definitions and notations. We will say that an operator T: $L_p(\Omega, \mathfrak{B}, \mu) \to L_p(\Omega, \mathfrak{B}, \mu)$ has the pointwise ergodic property (p.e.p.) if for every $f \in L_p(\Omega, \mathfrak{B}, \mu)$, $[n^{-1}\Sigma_{i=0}^{n-1} T^i f](t)$ is a.e. convergent in Ω. It is known that T has the p.e.p. in the following cases: for $1 < p < \infty$, $p \neq 2$ and T an isometry (for an invertible isometry, A. Ionescu-Tulcea [7] and for a general isometry, R. V. Chacon and S. A. McGrath [6]), for $1 < p < \infty$ and T a positive contraction (M. A. Akcoglu [1]), and for $1 < p < \infty$ and T an isomorphism such that T and T^{-1} are both power-bounded and positive (A. de la Torre [12]). D. L. Burkholder [5] constructed a contraction T: $L_2 \to L_2$ failing to have the p.e.p. Akcoglu and Sucheston [3] remarked that T can be geometrically dilated to an isometry on an L_2 space that fails to have the p.e.p.

In this paper we construct for each $1 < p < 2$ an isomorphism T_p: $L_p \to L_p$ failing to have the p.e.p. so that T_p and T_p^{-1} both are power-bounded. Moreover, T_p is the "same" operator for all $p \in (1, 2]$, i.e. if $p < r < 2$ then $T_p(L_r) \subset L_r$ and $T_p|_{L_r} = T_r$. In a remark following the construction we outline a modification of the construction so that we have, in addition, $[n^{-1}\Sigma_{i=0}^{n-1} T_p^i]$ (t) a.e. divergent in $[0, 1]$, for some $f \in L_p$.

An "operator" in this paper is always bounded and linear. Let X be a Banach space. If an operator P: $X \to X$ is idempotent ($P^2 = P$) then P is said to be a projection and $P(X)$ is said to be complemented in X. I: $X \to X$ denotes the identity. If A is a subset of X then $\overline{\text{conv}}\, A$ denotes the closed convex hull of A and $[A]$ its closed linear span. L_p is $L_p(0, 1)$ and the norm of $f \in L_p$ will be denoted by $\|f\|_p$ whenever $\|f\|$ is ambiguous. $f \in L_p$ will be treated as a function (defined a.e.). $f|_B$ is the restriction of f to B. Unless otherwise stated, the scalar field may be the real or complex one. The Hilbert space l_2 will be denoted sometimes by l_2^R or l_2^C to distinguish between the real and complex cases. The unit vector basis $\{e_i\}$ in l_2 is defined by $e_i = (0, \ldots, 0, 1, 0 \ldots)$, the 1 in the ith slot. A sequence $\{x_i\} \subset X$ is equivalent to the unit vector basis of l_2 if there is an isomorphism of l_2 and $[x_i]$

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carrying e_i to x_i for each i. If a is an element of an algebra and n is a positive integer then we put $A_n(a) = n^{-1}\Sigma_{i=0}^{n-1} a^i$. An operator T: $X \to X$ is power-bounded if $\sup_n \|T^n\| < \infty$.

It will be convenient to work in $L_p(0, 2)$ rather than in L_p $(= L_p(0, 1))$. When functions f, $g \in L_p$ are given we denote by $h = \langle f; g\rangle$ the function $h \in L_p(0, 2)$ defined by $h(t) = f(t)$ for $t \in [0, 1]$ and $h(t) = g(t - 1)$ for $t \in (1, 2]$. We also put $(f, g) = \int_0^1 f(t)\overline{g(t)}\, dt$.

2. Power-bounded operators. M. A. Akcoglu [2] modified an idea of D. L. Burkholder [5] to obtain the following.

LEMMA 1. *Let l_2^C be the complex Hilbert space, and for each n let R_n: $l_2^C \to l_2^C$ be the orthogonal projection onto $[e_{n+1}, e_{n+2}, \ldots]$. Let $\{\varepsilon_n\}$ be a sequence of positive real numbers. Then there exists a surjective isometry S: $l_2^C \to l_2^C$ and there is a sequence $0 < n_1 < n_2 < \cdots$ of integers so that $\|A_{n_k}(S) - R_k\| < \varepsilon_k$ for every k.*

COROLLARY 1. *Let l_2^R be the real Hilbert space, R_n: $l_2^R \to l_2^R$ defined similarly and $\{\varepsilon_n\}$ as in Lemma 1. Then there is a surjective isometry S_R: $l_2^R \to l_2^R$ and there is a sequence $0 < n_1 < n_2 \cdots$ of integers so that $\|A_{n_k}(S_R) - R_{2k}\| < \varepsilon_k$ for every k.*

PROOF. Let A: $l_2^C \to l_2^R$ be the real-linear map sending $(x_1, x_2, \ldots)$ to $(\operatorname{Re} x_1, \operatorname{Im} x_1, \operatorname{Re} x_2, \operatorname{Im} x_2, \ldots)$. Now put $S_R = ASA^{-1}$, the S from Lemma 1.

REMARK 1. Lemma 1 can be generalized so that it remains true for a Banach space X with a basis $\{x_i\}$: Let P_n: $X \to X$ be given by $P_n(\Sigma\, \alpha_i x_i) = \Sigma_{i<n}\, \alpha_i x_i$, $P_0 = 0$ and let $\{\varepsilon_n\}$ be as in the lemma. Then

(a) There exist a power-bounded isomorphism T: $X \to X$ and a sequence $0 < n_1 < n_2 < \cdots$ of integers so that $\|A_{n_k}(T) - (I - P_k)\| < \varepsilon_k$ for every k.

(b) If the basis $\{x_n\}$ is unconditional and the scalar field the complex numbers, we can make T and T^{-1} both power-bounded.

SKETCH OF THE PROOF. If $\{\lambda_n\}$ is a decreasing (increasing) sequence in [0, 1] and T: $X \to X$ is given by $T(\Sigma\, \alpha_i x_i) = \Sigma\, \lambda_i \alpha_i x_i$, then $T \in \overline{\operatorname{conv}}\{P_n\colon n = 0, 1, \ldots\}$ (respectively, $T \in \overline{\operatorname{conv}}\{I - P_n\colon n = 0, 1, \ldots\}$). Since the basis constant $K = \sup\|P_n\|$ is always finite, T is a well-defined operator and for every n, $\|T^n\| \leqslant K$ (respectively, $\|T^n\| \leqslant K + 1$). Now construct monotone sequences $0 < \lambda_1 < \lambda_2 < \cdots < 1$ and $0 < n_1 < n_2 < \cdots$ inductively, in a similar way to Akcoglu's construction. Note that if, for a given j, $A_n(\lambda_i) < \delta$ for $i \leqslant j$ and $A_n(\lambda_i) > 1 - \delta$ for $i \geqslant j + 1$ then $A_n(T) - (I - P_j)$ is in the set

$$\delta \cdot \operatorname{conv}\{P_0, \ldots, P_j\} + \delta \cdot \overline{\operatorname{conv}}\,\{I - P_i\colon i > j\}$$

and thus of norm less than $(2K + 1)\delta$.

Part (b) is proved in the same way as Akcoglu's lemma (with the λ_i's complex, $|\lambda_i| = 1$).

LEMMA 2. *Let A_n: $L_p \to L_p$ be a sequence of operators with $\Sigma\|A_n\| < \infty$. Then for every $f \in L_p$ the series $\Sigma(A_n f)(t)$ converges a.e. in* [0, 1].

PROOF. Follows easily from

$$\Sigma \int_0^1 |(A_n f)(t)|^p\, dt = \Sigma\, \|A_n f\|^p \leqslant \|f\|^p \Sigma\, \|A_n\|^p < \infty.$$

3. Construction of the operator. Let $\{r_n(t)\}$ be the Rademacher functions on $[0, 1]$ defined by $r_n(t) = \operatorname{sign} \sin 2^n \pi t$ $(n = 0, 1, \ldots)$ where sign $0 = 1$. The Rademacher functions span a complemented subspace of L_p for every $1 < p < \infty$, and the sequence $\{r_n\} \in L_p$ is equivalent to the unit vector basis of l_2 for every $1 < p < \infty$; i.e. there are positive constants A_p and B_p so that for every $\{a_n\} \in l_2$,

$$A_p\left(\sum |a_n|^2\right)^{1/2} \leqslant \left\|\sum a_n r_n\right\|_p \leqslant B_p\left(\sum |a_n|^2\right)^{1/2}$$

(Khinchin's inequality), and the map V: $L_p \to L_p$, defined by $Vf = \Sigma(f, r_n)r_n$ where $(f, r_m) = \int_0^1 f(t) r_n(t)\, dt$, is a (bounded) projection when $1 < p < \infty$ (see A. Pełczyński [11], Proposition 5). D. Menchoff [10] constructed an orthonormal sequence $\{g_n\}$ in L_2 and a sequence of real numbers $\{c_n\} \in l_2$ so that $\Sigma\, c_n g_n(t)$ is a.e. divergent in $[0, 1]$. Define $\{\rho_n\}$ and $\{y_n\}$ in $L_p(0, 2)$ by $\rho_n = \langle r_n; 0\rangle$, $y_{2n-1} = \langle 0; g_n\rangle$ and $y_{2n} = 0$. Let $\{a_n\} \in l_2$. Then for $p \in (1, 2]$,

$$\begin{aligned} A_p\left(\sum |a_n|^2\right)^{1/2} &\leqslant \left\|\sum a_n r_n\right\|_p \leqslant \left\|\sum a_n(\rho_n + y_n)\right\|_p \\ &\leqslant \left\|\sum a_n \rho_n\right\|_p + \left\|\sum a_n y_n\right\|_p \\ &\leqslant \left\|\sum a_n r_n\right\|_2 + \left\|\sum a_{2n-1} g_n\right\|_2 \leqslant 2\left(\sum |a_n|^2\right)^{1/2}. \end{aligned}$$

Thus $Be_i = \rho_i + y_i$ defines an isomorphism B: $l_2 \to [\rho_n + y_n]$. Let R: $L_p(0, 2) \to L_p$ be the restriction map $Rf = f|_{[0,1]}$ and let A: $[r_n] \to L_p(0, 2)$ be given by $A(r_n) = \rho_n + y_n$. A is an isomorphism (into). We put $U = AVR$. For every n,

$$U(\rho_n + y_n) = AVR(\rho_n + y_n) = AV(r_n) = A(r_n) = \rho_n + y_n.$$

Therefore U is a projection of $L_p(0, 2)$ onto $[\rho_n + y_n]$. For a.e. $t \in [1, 2]$ the series $\Sigma\, c_n(\rho_{2n-1} + y_{2n-1})(t) = \Sigma\, c_n g_n(t - 1)$ is divergent. Denote

$$f = \sum c_n(\rho_{2n-1} + y_{2n-1}) \in L_p(0, 2).$$

By Lemma 1 and Corollary 2, there is an isometry S: $l_2 \to l_2$ and a sequence $0 < n_1 < n_2 < \cdots$ of integers so that $\Sigma\|A_{n_k}(S) - R_{2k}\| < \infty$. Now put $T = BSB^{-1}U + I - U$. T is an isomorphism of $L_p(0, 2)$, T and T^{-1} both are power-bounded, and $[A_n(T)f](t)$ is a.e. divergent in $[1, 2]$:

$$\left[BR_{2k}B^{-1}Uf\right](t) = \sum_{i=k+1}^{\infty} c_i(\rho_{2i-1} + y_{2i-1})(t)$$

for every t and

$$\sum \left\|A_{n_k}(T) - \left(BR_{2k}B^{-1}U + I - U\right)\right\| < \infty.$$

Since $(I - U)f = 0$, we have the result using Lemma 2.

REMARK 2. A technically more involved construction gives, for $1 < p \leqslant 2$, an isomorphism T: $L_p(0, 2) \to L_p(0, 2)$ having, in addition, the property that $[A_n(T)f](t)$ is divergent a.e. in $[0, 2]$.

SKETCH OF THE CONSTRUCTION. Since $p > 1$, the Haar system $\{\phi_i\}$ is an unconditional basis of L_p (see, e.g., [9]). A simple modification of Kaczmarz' version [8] of Menchoff's construction gives an orthonormal sequence $\{g_i\}$ in L_2 and a sequence $\{c_i\} \in l_2$ with $\Sigma\, c_i g_i(t)$ a.e. divergent in $[0, 1]$ such that there are sequences of integers $0 < k_1 < k_2 < \cdots$ and $0 < m_1 < m_2 < \cdots$ satisfying for every i, $g_i \in [\phi_{k_{2i}+1}, \ldots, \phi_{k_{2i}}]$ and $r_{m_i} \in [\phi_{k_{2i+1}+1}, \ldots, \phi_{k_{2i+2}}]$. Now define a sequence $\{y_i\}$ in $L_p(0, 2)$ by

$$y_{4i-3} = \langle r_{m_{2i-1}}; g_i \rangle, \quad y_{4i-2} = \langle r_{m_{2i}}; 0 \rangle,$$
$$y_{4i-1} = \langle g_i; r_{m_{2i-1}} \rangle, \quad y_{4i} = \langle 0; r_{m_{2i}} \rangle.$$

Let $\{a_i\} \in l_2$. Since $1 < p \leqslant 2$,

$$\left\| \sum a_i y_i \right\| \geqslant \left\| \sum a_i y_i|_{[0,1]} \right\| \geqslant K_p \left\| \sum (a_{4i-3} r_{m_{2i-1}} + a_{4i-2} r_{m_{2i}}) \right\|$$
$$\geqslant K_p A_p \left[\sum \left([a_{r4i-3}]^2 + [a_{s4i-1}]^2 \right) \right]^{1/2}$$

where K_p is the unconditional constant of the Haar basis. Similarly,

$$\left\| \sum a_i y_i \right\| \geqslant K_p A_p \left[\sum \left(|a_{4i-1}|^2 + |a_{4i}|^2 \right) \right]^{1/2}.$$

Thus

$$\left\| \sum a_i y_i \right\| \geqslant 2^{-1/2} K_p A_p \left(\sum |a_i|^2 \right)^{1/2}.$$

It is easy to see that $\|\sum a_i y_i\| \leqslant 2^{1/p} (\sum |a_i|^2)^{1/2}$. Thus, $\{y_i\}$ is equivalent to the unit vector basis of l_2. Now define R_j: $L_p(0, 2) \to L_p$ $(j = 1, 2)$ by $R_1 f = f|_{[0,1]}$ and $[R_2 f](t) = f(t + 1)$ for $t \in [0, 1]$. Define A_j: $[r_{m_i}] \to [y_i]$ $(j = 1, 2)$ by

$$A_1 r_{m_{2i-1}} = y_{4i-3}, \quad A_1 r_{m_{2i}} = y_{4i-2},$$
$$A_2 r_{m_{2i-1}} = y_{4i-1}, \quad A_2 r_{m_{2i}} = y_{4i}.$$

Let V be the projection in L_p defined by $Vf = \Sigma (f, r_{m_i}) r_{m_i}$. The map P: $L_p(0, 2) \to L_p(0, 2)$, $P = A_1 V R_1 + A_2 V R_2$ is a projection onto $[y_i]$ (note that $V g_i = 0$ for all i). Put $f = \Sigma\, c_i (y_{4i-3} + y_{4i-1})$; then $\Sigma\, c_i (y_{4i-3} + y_{4i-1})(t)$ is a.e. divergent in $[0, 2]$ (to see this note that $\Sigma\, c_i r_{m_{2i-1}}(t)$ is a.e. convergent in $[0, 1]$–cf. [4]). The end of the construction is similar to the preceding one.

REMARK 3. The Haar basis $\{\phi_i\}$ has the following property: if $\Sigma\, \alpha_i \phi_i \in L_p$ then $\Sigma\, \alpha_i \phi_i(t)$ converges a.e. in $[0, 1]$ (see [4]). Using a similar idea to the one used in our construction we can construct an unconditional basis $\{x_i\}$ of L_p and a function $f = \Sigma\, \alpha_i x_i \in L_p$ with $\Sigma\, \alpha_i x_i(t)$ a.e. divergent in $[0, 1]$. Again, for reasons of convenience we consider $L_p(0, 2)$ instead of L_p. Using Pełczyński's decomposition method ([11]; see also [9]), it is easy to see that if P is as in Remark 2 then $(I - P) L_p(0, 2)$ is isomorphic to L_p and thus has an unconditional basis $\{z_i\}$. Now define $x_{2i-1} = y_i$, $x_{2i} = z_i$ and f as in Remark 2.

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