COMPOSING FUNCTIONS OF BOUNDED VARIATION

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Abstract. We find necessary and sufficient conditions on \( g \) such that \( f \circ g \) (resp. \( g \circ f \)) is a function of bounded variation for all \( f \) of bounded variation.

Let \( I = [0, 1] \), and let \( BV \) denote the set of functions \( f : I \to I \) of bounded variation. \( BV \) is not closed under composition (example: \( f \circ g \) where \( f(x) = \sqrt{x} \); \( g(0) = 0, g(x) = x^2 \sin^2(1/x) \) for \( x \neq 0 \)). Here we characterize those functions \( g \) whose right (resp. left) composition operation \( f \to f \circ g \) (resp. \( f \to g \circ f \)) preserves \( BV \).

If \( f : I \to \mathbb{R} \), define \( v(f, a, b) \) as the variation of \( f \) on the subinterval \([a, b]\), and \( v(f) = v(f, 0, 1) \). For a positive integer \( N \), let \( J_N = \{X \subseteq I: X \text{ can be expressed as a union of } N \text{ intervals} \} \) (where the intervals may be open or closed at either end and we allow singletons as degenerate closed intervals). Since any interval is a union of two subintervals, we see \( J_N \subseteq J_{N+1} \). A function \( f : I \to \mathbb{R} \) is said to be of \( N \)-bounded variation if \( f^{-1}([a, b]) \in J_N \) for all \([a, b] \subseteq \mathbb{R} \). Let \( BV(N) \) be the set of all functions \( f : I \to I \) of \( N \)-bounded variation, and \( BV'(N) \) the set of all bounded functions \( f : I \to \mathbb{R} \) of \( N \)-bounded variation.

**Lemma 1.** Every function in \( BV'(N) \) is of bounded variation.

**Proof.** Suppose \( h \in BV'(N) \) and \( |h(x)| < M \) for all \( x \in I \). We show \( v(h) < 4M(N + 1) \). If \( v(h) > 4M(N + 1) \), then there exists a partition \( \{x_0, \ldots, x_n\} \) of \( I \) with

\[
\sum_{i=1}^{n} |h(x_i) - h(x_{i-1})| > 4M(N + 1).
\]

Some \([a', b'] \subseteq [-M, M]\) with \( a' < b' \) is covered more than \( 2(N + 1) \) times by intervals \([h(x_{i-1}), h(x_i)]\) or \([h(x_i), h(x_{i-1})]\). At least \( N + 1 \) are of the type \([h(x_{i-1}), h(x_i)]\), say, so

\[
[a', b'] \subseteq \bigcap_{j=1}^{N+1} [h(y_j), h(z_j)]
\]

where each \((y_j, z_j) = (x_{i-1}, x_i)\) for some \( i \), and \( y_1 < z_1 < \cdots < y_{N+1} < z_{N+1} \). Then \( h(y_j) < a' < b' < h(z_j) \) for all \( j \). Take \( a \) between \( a' \) and \( b' \); take \( b \) greater than
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every \( h(z_j) \). Then \( h(y_j) \notin [a, b] \), \( h(z_j) \in [a, b] \) for all \( j \). Hence \( h^{-1}([a, b]) \) is union of no fewer than \( N + 1 \) intervals, and so is not in \( J_N \), a contradiction. \( \cdot \) \( h \) is of bounded variation.

**Lemma 2.** Let \( h_i : I \to I \), \( i = 1, 2, \ldots \), be functions which assume only the values 0 and 3\(^i\). If \( h = \sum_{i=1}^{\infty} h_i \), then \( \nu(h) > \frac{1}{6} \sum_{i=1}^{\infty} \nu(h_i) \). In particular, if some \( h_i \notin BV \), then \( h \notin BV \).

**Proof.** Let \( p \) be a discontinuity of \( h_i \) which is not a discontinuity of \( h_j \) for \( j < i \). Then \( p \) is a discontinuity of \( h \), contributing a jump of at least \( 3^{-i} - \sum_{j=1}^{i-1} 3^{-j} = \frac{1}{2} 3^{-i} \) in \( h \), but contributing no more than \( 2 \sum_{j=i}^{\infty} 3^{-j} = 3^{-i+1} \) to \( \sum \nu(h_j) \). If each \( h_i \in BV \), i.e. has only a finite number of discontinuities, the inequality follows. Otherwise some \( h_i \) has infinitely many discontinuities, and \( \nu(h) = \nu(h_i) = \infty \).

**Theorem 3.** For \( g : I \to I \), the composition \( f \circ g \) belongs to \( BV \) for all \( f \in BV \) iff \( g \in BV(N) \) for some \( N \).

**Proof.** **Sufficiency.** Suppose \( f : I \to R \) is increasing, and \( g \in BV(N) \). Then \( f \circ g \in BV(N) \) because \( f^{-1}([a, b]) = [f^{-1}(a), f^{-1}(b)] \), and hence \( (f \circ g)^{-1}([a, b]) \in J_N \) for all \( a, b \). Moreover, any \( f \in BV \) can be expressed as a difference of two increasing functions \( f = f_1 - f_2 \), and since \( f_1 \circ g, f_2 \circ g \in BV(N) \), we have \( f \circ g = f_1 \circ g - f_2 \circ g \in BV \), using Lemma 1.

**Necessity.** Suppose for all \( N \), \( g \in BV(N) \). Find intervals \( J_n \subseteq I \), \( n = 1, 2, \ldots \), such that \( g^{-1}(J_n) \notin J_n \), i.e. \( g^{-1}(J_n) \) cannot be expressed as a union of \( 3^n \) intervals. Define \( f_n = 3^{-n} \chi_{J_n} \), and \( f = \sum_{n=1}^{\infty} f_n \). Since \( \nu(f_n) < 2 \cdot 3^{-n} \), we have \( \nu(f) < \sum \nu(f_n) < 2 \sum 3^{-n} < \infty \), so \( f \in BV \). On the other hand, \( f_n \circ g = 3^{-n} \chi_{g^{-1}(J_n)} \), so \( \nu(f_n \circ g) > 2 \cdot 3^{-n} \cdot 3^n = 2 \). By Lemma 2, \( \nu(f \circ g) = \infty \), i.e. \( f \circ g \notin BV \).

**Theorem 4.** For \( g : I \to I \), the composition \( g \circ f \) belongs to \( BV \) for all \( f \in BV \) iff \( g \) satisfies a Lipschitz condition on \( I \).

**Proof.** **Sufficiency.** Suppose \( |g(y) - g(z)| < M|y - z| \) for all \( y, z \in I \), and suppose \( f \in BV \) with \( \nu(f) = L \). If \( P = \{x_0, \ldots, x_n\} \) is a partition of \( I \),

\[
\sum_P |\Delta(g \circ f)| < \sum_{i=1}^{n} M|f(x_i) - f(x_{i-1})| < LM,
\]

so \( g \circ f \in BV \).

**Necessity.** If \( g \) does not satisfy a Lipschitz condition on \( I \), there exist \( y_n'' < z_n'' \) with \( |g(y_n'') - g(z_n'')| > (n^2 + n)|y_n'' - z_n''| \) for any \( n = 1, 2, \ldots \). Since \( I \) is compact, there is a convergent subsequence \( \{y_n'\} \) of \( \{y_n''\} \). Say \( y_n' \to y \). Take a further subsequence \( \{y_n\} \) with \( |y - y_n| < (n + 1)^{-2} \). Take \( \{z_n\} \) to be the corresponding subsequence of \( \{z_n''\} \), and \( \delta_n = |y_n - z_n| \). We have \( |g(y_n) - g(z_n)| > (n^2 + n)\delta_n \), and hence \( \delta_n < (n^2 + n)^{-1} \).

Define \( f \) by \( f(0) = 0, f(1) = y_1 \), while on \([n + 1]^{-1}, n^{-1} \),

\[
f(x) = \begin{cases} y_n & \text{if } x - (n + 1)^{-1} \text{ is a multiple of } \delta_n, \\ z_n & \text{otherwise.} \end{cases}
\]
For a fixed \( n \), let \( m' = \delta_n^{-1}(n^2 + n)^{-1} \), and \( m = \max\{k \in \mathbb{Z}: k < m'\} \). Since \( m' > 1 \), we have \( m'/2 < m < m' \). Consider the partition \( P \) of \([(n + 1)^{-1}, n^{-1}]\):

\[
P = \left\{ (n + 1)^{-1} + \frac{1}{2} r \delta_n: r = 0, 1, \ldots, 2m \right\} \cup \left\{ n^{-1} \right\}.
\]

Then

\[
\sum_{P} |\Delta(g \circ f)| > 2m|g(y_n) - g(z_n)| > m'(n^2 + n)\delta_n = 1,
\]

so \( v(g \circ f, (n + 1)^{-1}, n^{-1}) > 1 \): whence \( g \circ f \notin \text{BV} \).

On the other hand, since \( f \) is a step function on \([(n + 1)^{-1}, n^{-1}]\),

\[
v(f, (n + 1)^{-1}, n^{-1}) = 2m|y_n - z_n| + |y_{n-1} - t_n| \quad \text{where } t_n = y_n \text{ or } z_n
\]

\[
< 2m'\delta_n + |y_{n-1} - y| + |y - y_n| + |y_n - z_n|
\]

\[
< 2(n^2 + n)^{-1} + n^{-2} + (n + 1)^{-2} + n^{-2} < 5n^{-2}.
\]

Since \( \sum n^{-2} \) converges, \( f \in \text{BV} \).

Chaika and Waterman [1] obtained a theorem analogous to our Theorem 4, but for certain classes larger than \( \text{BV} \). Interestingly, they found the Lipschitz condition to be necessary and sufficient in those cases as well.

References


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