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NON-(CA) ANALYTIC GROUPS AND GROUPS OF ROTATIONS

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ABSTRACT. It is known that every non-(CA) analytic group arises from the action of a vector group on a (CA) analytic group. We prove here that this action always involves a vector group of *rotations* acting on another *vector* group.

1. Introduction. For any analytic group G, the group $\mathrm{Aut}(G)$ of all continuous automorphisms of G is a Lie group in the generalized compact-open topology. (See, for example, p. 40 of [2].) G is a (CA) *analytic group* if the group of inner automorphisms of G is closed in $\mathrm{Aut}(G)$.

Zerling [7, p. 182] has shown that every non-(CA) analytic group arises from the action of a vector group on a (CA) analytic group. The examples in [3], [4], [6], and [7], however, take a much less general form: all involve the action of a vector group of *rotations* on another *vector* group. In the present article we show that this is an essential feature of all non-(CA) analytic groups, rather than an accidental property of the particular examples in the literature. Although we will confine our attention to simply-connected, solvable analytic groups, this restriction does not materially affect the scope of our result. For an analytic group G is (CA) if and only if its simply-connected covering group G' is (CA), since G is the quotient of G' by a central subgroup. Moreover, Van Est [5, pp. 559–561] has demonstrated that G is (CA) if and only if its radical is (CA).

Before stating our result precisely, we must introduce some notation. First we describe in more detail the principal result in [7]. If G is a non-(CA) analytic group, then G has the form $M \circledS V$, where M is a (CA) analytic group and V is a vector subgroup of $\mathrm{Aut}(M)$ whose closure in $\mathrm{Aut}(M)$ is compact. We will say that $G = M \circledS V$ is a *standard decomposition* of G.

Now let θ be an element of the set $\mathbf{R}$ of real numbers. We will let R_θ denote the 2-by-2 matrix which corresponds to the rotation of $\mathbf{R}^2$ through a counterclockwise angle of θ. If n is an even natural number, let T_n be the subgroup of $\mathrm{Gl}(n, \mathbf{R})$ consisting of all matrices of the form shown in Figure 1, where $k = n/2$ and $\theta_1, \ldots, \theta_k \in \mathbf{R}$. If $n > 1$ is odd, we define a homomorphism $h\colon \mathrm{Gl}(n-1, \mathbf{R}) \to \mathrm{Gl}(n, \mathbf{R})$ by

$$h(A) = \begin{vmatrix} A & 0 \\ 0 & 1 \end{vmatrix}, \qquad A \in \mathrm{Gl}(n-1, \mathbf{R}),$$

and then let $T_n = h(T_{n-1})$. We can now proceed to our theorem.

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$$\begin{vmatrix} R_{\theta_1} & & & 0 \\ & R_{\theta_2} & & \\ & & \ddots & \\ 0 & & & R_{\theta_k} \end{vmatrix}$$

FIGURE 1

THEOREM. *Let G be a simply-connected, solvable, non-*(CA) *analytic group with standard decomposition $G = M \circledS V$. Let G_1 and G_2 be, respectively, the derived groups of G and G_1. Then $G_1 \subseteq M$ and $(G_1/G_2) \circledS V$ is a non-*(CA) *analytic group which is isomorphic, both topologically and algebraically, with $\mathbf{R}^n \circledS W$, where $n \geqslant 4$ is the dimension of G_1/G_2 and W is a vector subgroup of T_n whose closure in T_n is compact.*

2. Proof of theorem. Since V is abelian and G is not, it is clear that $G_1 \subseteq M$ and $G_1 \neq G_2$. That G_1 and G_1/G_2 are simply-connected analytic groups follows from the fact that G is simply-connected and G_1 and G_2 are normal in G [**2**, pp. 135–136]. Since G_1/G_2 is also abelian, it follows that G_1/G_2 is isomorphic to $\mathbf{R}^n$ for some $n > 0$.

We claim that V acts effectively on the vector group G_1/G_2. From Theorem 3.2 of [7] we know that V acts effectively on G_1 and may thus be regarded as a vector subgroup of $\mathrm{Aut}(G_1)$ with compact closure S. To establish our claim, we will show that S acts effectively on G_1/G_2. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be the Lie algebras of G_1 and G_2, respectively. The isomorphism of $\mathrm{Aut}(G_1)$ with $\mathrm{Aut}(\mathfrak{g}_1)$ defines a representation of S into $\mathrm{Gl}(\mathfrak{g}_1)$ which must be semisimple because S is compact. Therefore there exists an S-invariant subspace (not necessarily a subalgebra) $\mathfrak{m}$ of $\mathfrak{g}_1$ such that $\mathfrak{g}_1 = \mathfrak{g}_2 \oplus \mathfrak{m}$. If $s \in S$ induces the identity automorphism on G_1/G_2, then S acts as the identity not only on $\mathfrak{m}$ but also on the Lie algebra $\mathfrak{a}$ generated by $\mathfrak{m}$. An exercise in [**1**, Exercise 4, p. 91], however, shows that $\mathfrak{a} = \mathfrak{g}_1$, so that s must be the identity on all of G_1. This demonstrates that S acts effectively on G_1/G_2 (and, incidentally, that $n \geqslant 2$).

To finish the proof, let us choose some specific isomorphism of G_1/G_2 with $\mathbf{R}^n$. Regarding V as a nonclosed vector subgroup of $\mathrm{Gl}(n, \mathbf{R})$, we know that V is contained in a maximal torus T of $\mathrm{Gl}(n, \mathbf{R})$. Examination of the centralizer of T_n in $\mathrm{Gl}(n, \mathbf{R})$ reveals that T_n is also a maximal torus in $\mathrm{Gl}(n, \mathbf{R})$ and is therefore conjugate to T [**2**, pp. 152, 181]. Let A be an element of $\mathrm{Gl}(n, \mathbf{R})$ such that

$T_n = ATA^{-1}$, and let $W = AVA^{-1}$. It is easy to verify that the function f: (G_1/G_2) Ⓢ $V \to \mathbf{R}^n$ Ⓢ W defined by $f(x, v) = (A(x), AvA^{-1})$ for $x \in G_1/G_2$, $v \in V$, is an isomorphism of Lie groups. Since W is a nonclosed vector subgroup of T_n, it follows immediately that $n \geqslant 4$ and that neither $\mathbf{R}^n$ Ⓢ W nor (G_1/G_2) Ⓢ V is (CA).

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