In this paper we prove an irreducibility theorem for representations in a degenerate principal series for the real symplectic groups $\text{Sp}(n, \mathbb{R})$. These representations $T_\chi$ are induced from unitary characters $\chi$ of a certain maximal parabolic subgroup $H$. Here is the main result:

**Theorem.** The representation $T_\chi$ is irreducible for every nontrivial character $\chi$ of $H$. If $\chi$ is the trivial character then $T_\chi$ is the direct sum of two irreducible subrepresentations.

It should be noted that the analogous result over the complex field is proved by K. I. Gross in [3], and our methods are analogous. What is surprising in the real case is that, although the index of squares in the real field is two and consequently there are twice as many "orbits" in the real case as in the complex case (cf. below), there is no additional reducibility.

Here is a brief sketch of the steps in the proof. (1) Realize $T_\chi$ to act in the space $L^2(Z)$, where $Z$, the nilpotent radical for the parabolic subgroup $H$ "opposite" to $\widetilde{H}$, is a 2-step nilpotent subgroup of $\text{Sp}(n, \mathbb{R})$. (2) Apply the Plancherel transform $\mathcal{P}$ of $L^2(Z)$ and consider the unitarily equivalent representation $\hat{T}_\chi = \mathcal{P}T_\chi\mathcal{P}^{-1}$ of $\text{Sp}(n, \mathbb{R})$. It is the transformed representation $\hat{T}_\chi$ that we will actually analyze. (3) Let $\mathcal{B}(T_\chi)$ denote the von Neumann algebra of the representation and let $\mathcal{B}'(T_\chi)$ denote the commuting algebra. According to Schur's Lemma, we must determine whether $\mathcal{B}'(T_\chi)$ consists solely of scalar multiples of the identity operator. Thus, let $B \in \mathcal{B}'(T_\chi)$, so $\mathcal{P}B\mathcal{P}^{-1} \in \mathcal{B}'(\hat{T}_\chi)$. (4) When $\hat{T}_\chi$ is restricted to $H$, we can explicitly determine the commuting algebra $\mathcal{B}'(\hat{T}_\chi|_H)$ of the restriction of $\hat{T}_\chi$ to $H$. It is 4-dimensional and is spanned by a certain set of operators $\{I, \mathcal{P}J\mathcal{P}^{-1}, \mathcal{P}I_a\mathcal{P}^{-1}, \mathcal{P}J_a\mathcal{P}^{-1}\}$. Here $I$ (the identity), $J$, $I_a$, and $J_a$ are operators back on the original representation space $L^2(Z)$; and of course, they form a basis for the commuting algebra $\mathcal{B}'(T|_H)$. (5) Since $H$ is a maximal subgroup of $\text{Sp}(n, \mathbb{R})$, it is enough to analyze just one element of the group outside of $H$. For
this purpose, a "Weyl reflection" \( p \) is a propitious choice. Thus, for \( B = \alpha I + \beta J + \gamma I_a + \delta J_a \), we know that \( B \) is in \( \mathcal{G}(T_\chi) \) if and only if \( BT_\chi(p) = T_\chi(p)B \). Finally, one shows, using a Mellin-type analysis on \( L^2(Z) \), that this condition is satisfied for nontrivial \( \chi \) if and only if \( \beta = \gamma = \delta = 0 \); the condition is satisfied for trivial \( \chi \) if and only if \( \gamma = \delta = 0 \).

With reference to [3], what is novel in the real case is the appearance of the convolution operators \( I_a \) and \( J_a \) on the nilpotent group, and a more intricate Mellin analysis in the last step. In what follows we refer the reader to [3], or [2], for details of proofs that are the same over the real field as for the complex field. We include the details where the real field introduces significant variations, for example in the Mellin analysis.

The author is indebted to K. I. Gross for many helpful comments and suggestions.

1. Reduction with respect to the opposite parabolic. Let \( n > 1 \) and \( m = n - 1 \). Represent the real symplectic group \( \text{Sp}(n, \mathbb{R}) \) as

\[
\Sigma_n = \left\{ g \in \mathbb{R}^{2n \times 2n} : gpg' = p, \quad p = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \right\},
\]

where \( g' \) denotes the transpose of the matrix \( g \). Define the following subgroups of \( \Sigma_n \):

\[
A = \left\{ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} : \alpha \in \mathbb{R}, \alpha \neq 0 \right\},
\]

\[
S = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s_{11} & 0 & s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & s_{21} & 0 & s_{22} \end{pmatrix} : s = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \in \Sigma_m \right\},
\]

\[
Z = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -y' & I & 0 & 0 \\ t & x & 1 & y \\ x' & 0 & 0 & I \end{pmatrix} : t \in \mathbb{R}, x, y \in \mathbb{R}^{1 \times m} \right\},
\]

\[
\overline{H} = Z'SA.
\]

It is useful to identify \( A \) with \( \mathbb{R}^* \) in the obvious way and use \( \alpha \) to denote both an element of \( \mathbb{R}^* \) and the corresponding element of \( A \). Similarly, we use \( s \) for an element of \( \Sigma_m \) and the corresponding element of \( S \). Elements of \( Z \) are denoted \( z \) or \( (x, y, t) \).

The degenerate principal series representation \( T_\chi \) induced by a unitary character \( \chi \) of \( \overline{H} \) is given (see [2, (1.1)]) by

\[
(1.1) \quad T_\chi(g)f(z) = \delta_\overline{H}(zg)^{-1/2} \chi(zg)f(zg)
\]

for \( g \in \Sigma_n \) and \( f \in L^2(Z) \), where \( \delta_\overline{H} \) is the modular function of \( \overline{H} \). By an easy computation, \( \delta_\overline{H}(h) = |\alpha|^{-2m-2} \) for \( h = z's\alpha \in \overline{H} \), and every unitary character has
the form \( \chi(h) = |\alpha|^{\alpha} \) for some \( \alpha \in \mathbb{R} \). If \( g = zs\alpha \) is an element of the opposite parabolic subgroup \( H = ZSA \) and \( z \in \mathbb{Z} \) then the action of \( \Sigma_n \) on \( Z \) occurring in (1.1) is given by \( \xi g = \xi(zs\alpha) = \alpha^{-1}z^{-1}\xi z\alpha \). In this case, (1.1) becomes
\[
T_{\xi}(zs\alpha)f(\xi) = |\alpha|^{m+1+is}\xi(\alpha^{-1}z^{-1}\xi z\alpha), \quad \text{for } f \in L^2(\mathbb{Z}).
\]

The Plancherel transform of \( L^2(\mathbb{Z}) \) involves the dual space \( \Lambda \) of \( Z \). Let \( V = \mathbb{R}^{1 \times m} \) and \( (y|\xi) = y'\xi' \) for any \( x,y \in V \). Identify \( \Lambda \) with \( \mathbb{R}^* \) as follows: to \( \lambda \in \mathbb{R}^* \), associate the irreducible unitary representation \( \hat{\Lambda} : Z \to \mathcal{U}(L^2(V)) \) given by
\[
\hat{\lambda}(z)\phi(u) = \exp\left[i\lambda \left( t - (y|2u + x) \right) \right] \phi(u + x),
\]
where \( z = (x,y,t) \in \mathbb{Z} \) and \( \phi \in L^2(V) \). We regard \( \Lambda \) as the set \( \{ \hat{\lambda} : \lambda \in \mathbb{R}^* \} \) supplied with the Lebesgue measure of \( \mathbb{R}^* \).

Let \( HS(L^2(V)) \) denote the Hilbert space of Hilbert-Schmidt operators on \( L^2(V) \). The Plancherel transform \( \mathcal{P} : L^2(\mathbb{Z}) \to L^2(\Lambda, HS(L^2(V)), d\mu(\lambda)) \) is defined for \( f \) in a dense subspace of \( L^2(\mathbb{Z}) \) by \( \mathcal{P}f(\lambda) = \int_Z f(z)\hat{\lambda}(z) \, dz \). The Plancherel measure on \( \Lambda, d\mu(\lambda) = (2\pi)^{-\frac{1}{2}m} |\lambda|^{m} \, d\lambda \), makes \( \mathcal{P} \) an isometry.

The following analogue of Theorem 2 of [3] introduces representations of \( S \) and \( A \) which play a role in the reduction of \( T|_H \):

**Theorem 1.4.** For \( \hat{\lambda} \in \Lambda, s \in S, \) and \( \alpha \in A \) define the representation \( sa\hat{\lambda} \) of \( Z \) by \( (sa\hat{\lambda})(z) = \hat{\lambda}(\alpha^{-1}z^{-1}s\alpha) \). Fix \( \lambda \in \mathbb{R}^* \); then (i) \( sa\hat{\lambda} \) is unitarily equivalent to \( (\lambda\alpha^2)^s \). (ii) There exists a unique multiplier representation \( \hat{\lambda} \) of \( S \) which satisfies \( \hat{\lambda}(s^{-1}z) = \hat{\lambda}(s)^{-1}\hat{\lambda}(z)\hat{\lambda}(s) \) for all \( z \in \mathbb{Z}, s \in S \). (iii) Let \( D \) be the representation of \( A \) in \( L^2(V) \) given by \( D(\alpha)\phi(u) = |\alpha|^{m/2}\phi(\alpha u) \). Then \( D \) is the unique representation of \( A \), up to tensor product with characters, such that \( \hat{\lambda}(\alpha^{-1}z\alpha) = D(\alpha)^{-1}(\lambda\alpha^2)^s D(\alpha) \) for all \( z \in \mathbb{Z} \). Also, \( (\lambda\alpha^2)^s = D(\alpha)\hat{\lambda}(s)D(\alpha)^{-1} \) for all \( s \in S \) so that \( \hat{\lambda} \) and \( (\lambda\alpha^2)^s \) are unitarily equivalent. (iv) The operators \( D(\pm 1) \) form a basis for the commuting algebra \( \mathcal{O}(\hat{\lambda}) \).

We remark that the representation \( \hat{\lambda} \) of \( S \cong \text{Sp}(n - 1, \mathbb{R}) \) is a variant of the oscillator, or metaplectic representation.

Fix a unitary character \( \chi \) of \( H \) and let \( T = T_x \). Define \( \hat{T} = \mathcal{P}T\mathcal{P}^{-1} \); then \( \hat{T} \) is a representation of \( \Sigma_n \) unitarily equivalent to \( T \) and acting in \( \mathcal{H} = L^2(\Lambda, HS(L^2(V)), d\mu(\lambda)) \). For \( f \) in a certain dense subspace of \( L^2(\mathbb{Z}) \) and \( \hat{f} = \mathcal{P}f \), \( \hat{T}(zs\alpha)\hat{f} \) is given by
\[
\hat{T}(zs\alpha)\hat{f}(\lambda) = |\alpha|^{-m-1+is}\hat{\lambda}(s)D(\alpha)\hat{f}(\alpha^{-2}\lambda)D(\alpha)^{-1}\hat{\lambda}(s)^{-1}\hat{\lambda}(z)^{-1}
\]
for any \( zs\alpha \in H \).

Analysis of this formula as in [2, Theorem 3.3] and [3, Theorem 6] yields

**Theorem 1.6.** The commuting algebra \( \mathcal{O}(\hat{T}|_H) \) is the algebra of all operators on \( \mathcal{H} \) of the form \( B = \int_{\Lambda} B(\lambda) \otimes I \, d\mu(\lambda) \), where \( \lambda \to B(\lambda) \) is a continuous mapping of \( \Lambda \) into \( \mathcal{U}(L^2(V)) \) satisfying (i) \( B(\lambda) \in \mathcal{O}(\hat{\lambda}) \) for all \( \lambda \in \Lambda = \mathbb{R}^* \), and (ii) \( B(\pm \alpha) = D(\alpha)B(\pm 1)D(\alpha)^{-1} \) for all \( \alpha \in A = \mathbb{R}^* \).

Let \( \Lambda^+ = (0, \infty) \) and \( \Lambda^- = (-\infty, 0) \) be the subsets of \( \Lambda \) determined by the identification with \( \mathbb{R}^* \). The two preceding theorems combine to give the basis...
\{I, J, I_a, J_a\} for the 4-dimensional algebra \( \mathfrak{A}'(T|_H) \), where

\[
\hat{I} = \text{the identity operator on } \mathfrak{H},
\]

\[
\hat{J} = \int_{\Lambda} D(-1) \otimes 1 \, d\mu(\lambda),
\]

(1.7)
\[
\hat{J}_a = \int_{\Lambda^+} I \otimes 1 \, d\mu(\lambda) - \int_{\Lambda^-} I \otimes 1 \, d\mu(\lambda), \quad \text{and}
\]

\[
\hat{J}_a = \int_{\Lambda^+} D(-1) \otimes 1 \, d\mu(\lambda) - \int_{\Lambda^-} D(-1) \otimes 1 \, d\mu(\lambda).
\]

The corresponding basis \( \{I, J, I_a, J_a\} \) can be found by applying the inverse transform \( B = \mathfrak{F}^{-1} \hat{B} \mathfrak{F} \). Define the following unitary operators on \( L^2(Z) \):

\[
Kf(x, y, t) = |t|^m f(y, -tx, t),
\]

(1.8)
\[
\mathfrak{F}_1 f(x, y, t) = (2\pi)^{-m/2} \int_{V} f(u, y, t) \exp i(x|u) \, du,
\]

\[
\mathfrak{F}_2 f(x, y, t) = (2\pi)^{-m/2} \int_{V} f(x, u, t) \exp i(y|u) \, du,
\]

\[
\mathfrak{F}_3 f(x, y, t) = (2\pi)^{-1/2} \int_{R} f(x, y, \lambda) \exp i\lambda \, d\lambda,
\]

\[
M_3 f(x, y, t) = (\text{sgn } t) f(x, y, t),
\]

for \( f \) in a suitable dense subset of \( L^2(Z) \). Note that \( \mathfrak{F}_1, \mathfrak{F}_2, \) and \( \mathfrak{F}_3 \) are the partial (Euclidean) Fourier transforms. In analogy to [3, p. 415] one can prove the following:

**Theorem 1.9.** The commuting algebra \( \mathfrak{A}'(T|_H) \) is 4-dimensional with basis \( \{I, J, I_a, J_a\} \), where \( J = \mathfrak{F}_3^{-1} K \mathfrak{F}_1 \mathfrak{F}_2 \mathfrak{F}_3 \), \( I_a = \mathfrak{F}_3^{-1} M_3 \mathfrak{F}_1 \mathfrak{F}_2 \mathfrak{F}_3 \), and \( J_a = \mathfrak{F}_3^{-1} M_3 K \mathfrak{F}_1 \mathfrak{F}_2 \mathfrak{F}_3 \).

In reference to our introductory remarks for the complex group \( \text{Sp}(n, \mathbb{C}) \), the commuting algebra \( \mathfrak{A}'(T|_H) \) in the complex case is just 2-dimensional. The reason for this difference involves the action of \( \Lambda \) on \( \Lambda \) given by \( \lambda \rightarrow \alpha^2 \lambda \). Condition (ii) of Theorem 1.6 states that under this action, \( B(\lambda) \) is uniquely determined on each orbit in \( \Lambda \) by evaluation at a fixed representative \( \lambda_0 \) in the orbit. In the real case there are two orbits allowing for two independent choices, say \( B(1) \) and \( B(-1) \), from \( \mathfrak{A}'(\lambda) \). But in the complex case, the action is transitive so that \( B \) is uniquely determined by the choice of \( B(1) \) from the 2-dimensional algebra \( \mathfrak{A}'(\lambda) \). This explains our earlier comment regarding the index of squares in the field.

Although the description of \( \mathfrak{A}'(T|_H) \) given in Theorem 1.9 is sufficient for the needs of the next section, it is interesting to view the operators \( J, I_a, \) and \( J_a \) as convolutions on the nilpotent group \( Z \) by generalized functions. Let \( \mathfrak{S} \) be the space of test functions consisting of all infinitely differentiable, rapidly decreasing functions on \( Z \). Let \( \mathfrak{S}' \) be the space of continuous linear functionals on \( \mathfrak{S} \); i.e., the generalized functions or tempered distributions on \( Z \). As is customary, for a locally integrable function \( \phi \) on \( Z \), the formula \( (\phi, f) = \int_{Z} \phi(z) f(z) \, dz \) \( (f \in \mathfrak{S}) \) defines an element of \( \mathfrak{S}' \) which is also called \( \phi \).
Let $A \in L^\infty(L^2(Z))$ and suppose $A : \mathbb{S} \rightarrow C(Z)$. Define, for each $f \in \mathbb{S}$, $(\phi_A, f) = Af(e)$, where $f^*(z) = f(z^{-1})$ and $e$ is the identity of $Z$; then $\phi_A$ is a generalized function. If we suppose also that $A \tau(z) = \tau(z)A$ for all right translations $\tau(z)$ of $L^2(Z)$ then $A$ is the generalized convolution given by $Af(z) = \phi_A * f(z) = (\phi_A, \tau(z)f)^*$, for $f \in \mathbb{S}$. By the usual abuse of notation we can write this as $Af(z) = \int_Z \phi_A(\xi)(\xi^{-1}z) \, d\xi$.

The operators $J, I_a, and J_a$ are generalized convolutions. To get some feeling for the generalized functions $\phi_J, \phi_I, and \phi_J$, consider the operators $A_1 = \mathfrak{F}^{-1}K \mathfrak{F}_1 \mathfrak{F}_2 = J \mathfrak{F}^{-1}, A_2 = \mathfrak{F}^{-1}M_3 = I_a \mathfrak{F}^{-1}, and A_3 = \mathfrak{F}^{-1}M_3K \mathfrak{F}_1 \mathfrak{F}_2 = J_a \mathfrak{F}^{-1}$. By direct computations,

$$
\phi_{A_1}(x, y, t) = (2\pi)^{-m-1/2} |t|^{-m},
$$

(1.10)  
\[ \phi_{A_2}(x, y, t) = (2\pi)^{-1/2} \delta(x, y)(sgn t), \text{ and} \]

$$
\phi_{A_3}(x, y, t) = (2\pi)^{-m-1/2} |t|^{-m},
$$

where the formula for $\phi_{A_j}$ has the meaning $(\phi_{A_j}, f) = \int_Z (2\pi)^{-1/2} (sgn t) f(0, 0, t) \, dt$, for any $f \in \mathbb{S}$.

If $\phi \in \mathbb{S}'$, define $\mathfrak{F}^{-1}\phi$ to be the generalized function given by $(\mathfrak{F}^{-1}\phi, f) = (\phi, \mathfrak{F}_3 f)$ $(f \in \mathbb{S})$. Since $\mathfrak{F}_3: \mathbb{S} \rightarrow \mathbb{S}$, the definition makes sense.

**Theorem 1.11.** The operators $J, I_a, and J_a$ are convolutions by the generalized functions $\mathfrak{F}^{-1}\phi_J, \mathfrak{F}^{-1}\phi_I, and \mathfrak{F}^{-1}\phi_J$, respectively where $\phi_J, \phi_I, and \phi_J$ are given by (1.10).

**Proof.** $(\phi_J, f) = (Jf)^*(e) = (A_1 \mathfrak{F}_3 f)^*(e) = (\phi_{A_1}, \mathfrak{F}_3 f) = (\mathfrak{F}^{-1}\phi_{A_1}, f)$. Similarly for $I_a$ and $J_a$.

2. Mellin analysis. It is known that $H = ZSA$ is a maximal subgroup of $\Sigma_n$ and, therefore, $\Sigma_n$ is generated by $H \cup \{p\}$, where $p = [\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}]$. It follows that $B \in \mathcal{C}'(T)$ if and only if $B \in \mathcal{C}'(T|_H)$ and $B$ commutes with $T(p)$. By Theorem 1.9 the elements of $\mathcal{C}'(T|_H)$ have the form

$$
B = \alpha I + \beta J + \gamma I_a + \delta J_a \quad (\alpha, \beta, \gamma, \delta \in \mathbb{C}).
$$

In this section we will show that an operator of the form (2.1) commutes with $T(p)$ if and only if one of the two following cases occurs: (i) $\chi$ is a nontrivial character and $\beta = \gamma = \delta = 0$ or (ii) $\chi$ is the trivial character and $\gamma = \delta = 0$.

Let the unitary character $\chi$ of $H$ be given by $\chi(h) = |\alpha|^\alpha$. From (1.1) the operator $T(p)$ is given by

$$
T(p)f(x, y, t) = |t|^{-m+1-ia}f(-t^{-1}y, t^{-1}x, -t^{-1}),
$$

for $f \in L^2(Z)$. Define unitary operators $T_3$ and $B_3(\chi)$ on $L^2(Z)$ by $T_3f(x, y, t) = |t|^{-1}f(x, y, -t^{-1})$, $B_3(\chi)f(x, y, t) = |t|^{|\alpha|}f(x, y, t)$. Then it is easy to verify that

$$
T(p) = B_3(\chi)T_3K,
$$

(2.4)

where $K$ is defined by (1.8).}

All of the relevant operators $T(p), J, I_a,$ and $J_a$ are now expressed in terms of $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, M_3, K, T_3,$ and $B_3(\chi)$. The notations for these latter operations have been
chosen, as in [3], so that operators with different subscripts commute. In order to consider a Mellin-type transform of these operators, we need to restrict them to certain subspaces of $L^2(Z)$ which transform under characters of a finite group; namely, the group $G = \{ 2^{m+1} \mathbb{Z} \}$ under multiplication. View the elements of $G$ as triples $(e_1, e_2, e_3)$, where $e_1$ and $e_2$ are $m \times m$ diagonal matrices with $\pm 1$ in each diagonal position and $e_3 = \pm 1$. Let $\hat{G}$ be the group of characters of $G$, let $\eta \in \hat{G}$, and define $L^2(Z)^\eta$ to be the subspace of $L^2(Z)$ consisting of all $f$ such that $f(xe_1, ye_2, te_3) = \eta(e_1, e_2, e_3)f(x, y, t)$ for all $(e_1, e_2, e_3) \in G$. It is not difficult to see that $L^2(Z) = \sum_{\eta \in \hat{G}} L^2(Z)^\eta$.

Each of the subspaces $L^2(Z)^\eta$ is invariant under the operations $T_1, T_2, T_3$ and $B_3(\chi)$. However, the operator $M_3$ permutes the subspaces sending $L^2(Z)^\eta$ to $L^2(Z)^{\eta'}$, where $e(e_1, e_2, e_3) = e_3$. The operator $K$ also permutes the subspaces. It sends $L^2(Z)^\eta$ to $L^2(Z)^{\eta'}$, where $\mu(e_1, e_2, e_3) = \eta(e_3 e_2, e_3 e_1, e_3)$. We shall call an element of $L^2(Z)^\eta$ a permutation operator if it permutes the set $\{ L^2(Z)^\eta : \eta \in \hat{G} \}$.

Thus, $T_1, T_2, T_3, B_3(\chi), M_3$, and $K$ are all permutation operators. Observe that:

(a) the permutations $M_3$ and $K$ have order 2 and (b) the permutations $M_3$ and $K M_3$ (or $M_3 K$) have no fixed points in the set $\{ L^2(Z)^\eta : \eta \in \hat{G} \}$.

Suppose that $aI + \beta J + \gamma L + \delta J \in \mathfrak{G}(T)$, with $\beta$, $\gamma$, $\delta$ not all zero, and let $f \in L^2(Z)^{\eta'}$. Then $\beta J + \gamma L + \delta J \in \mathfrak{G}(T)$ and, in particular,

$$\beta J T(p) f + \gamma L T(p) f + \delta J T(p) f = \beta T(p) J f + \gamma T(p) L f + \delta T(p) J f. \tag{2.5}$$

According to the preceding discussion, among the 6 terms in (2.5), only $\beta J T(p) f$ and $\beta T(p) J f$ are in $L^2(Z)^{\eta'}$; the other terms are in other subspaces of $L^2(Z)$. By a direct sum argument, then $\beta J T(p) f = \beta T(p) J f$. If $\beta \neq 0$ then it follows that $J \in \mathfrak{G}(T)$. Suppose $\beta = 0$ and $J \notin \mathfrak{G}(T)$; then $\gamma L + \delta J \in \mathfrak{G}(T)$ and hence $(\gamma L + \delta J)^2 = (\gamma^2 + \delta^2)I + 2\gamma \delta J \in \mathfrak{G}(T)$, which implies $2\gamma \delta = 0$. Thus, either $L \in \mathfrak{G}(T)$ or $J \in \mathfrak{G}(T)$. The result is that the existence of a nonscalar element of $\mathfrak{G}(T)$ implies that $J$, $L$, or $J$ is in $\mathfrak{G}(T)$. Therefore, it suffices to determine separately whether $J$, $L$, or $J$ commutes with $T(p)$.

In order to deal with these commutativity questions, we introduce a Mellin-type transform $\mathfrak{M}$ on $L^2(Z)$. For $x = [x_1, x_2, \ldots, x_m]$ and $u = [u_1, u_2, \ldots, u_m] \in V$, let $u^{-1/2}$ and $u^{ix}$ be defined by:

$$u^{-1/2} = u_1^{-1/2} \cdot u_2^{-1/2} \cdot \cdots \cdot u_m^{-1/2} \quad \text{and} \quad u^{ix} = u_1^{ix_1} \cdot u_2^{ix_2} \cdot \cdots \cdot u_m^{ix_m}. \tag{2.6}$$

For $f$ in a certain dense subspace of $L^2(Z)$, we define

$$\mathfrak{M} f(x, y, t) = c \int_0^\infty \cdots \int_0^\infty u^{-1/2} v^{-1/2} \lambda^{-1/2} f(u, v, \lambda) u^{ix} v^{iy} \lambda^it \ du \ dv \ d\lambda,$$

where $c = (2\pi)^{-(2m+1)/2}$. From the ordinary Mellin transform $\mathfrak{M}$ on $L^2(\mathbb{R}^+)$, it can easily be shown that (2.6) defines a surjective, bounded linear operator on $L^2(Z)$. Denote by $\mathfrak{M}_\eta$ the restriction of $\mathfrak{M}$ to the subspace $L^2(Z)^\eta$. Then $\mathfrak{M}_\eta : L^2(Z)^\eta \to L^2(Z)^\eta$ is a bijection, so we have a bounded linear transformation $\mathfrak{M}_\eta^{-1} : L^2(Z)^\eta \to L^2(Z)^\eta$ satisfying $\mathfrak{M}_\eta^{-1} \mathfrak{M}_\eta = I$, and $(\mathfrak{M}_\eta^{-1} \mathfrak{M}_\eta)_\eta = \mathfrak{M}_\eta^{-1} \mathfrak{M}_\eta = I_\eta$, where $I_\eta$ is the identity on $L^2(Z)^\eta$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Suppose that $X$ is a permutation operator. For fixed $\eta \in \hat{G}$, we define the $\eta$-transform of $X$, denoted $X^\eta$, by

\begin{equation}
X^\eta = \mathcal{R}_X \mathcal{R}_\eta^{-1}.
\end{equation}

Then $X^\eta \in L(L^2(\mathbb{Z}))$. The value of this transform in our context is suggested by the following two lemmas, the proofs of which are straightforward.

**Lemma 2.8.** Let $X$ and $Y$ be permutation operators and suppose $Y$ sends $L^2(\mathbb{Z})$ to $L^2(\mathbb{Z})^\eta$.

(i) $(XY)^\eta = X^\eta Y^\eta$, and

(ii) $(Y^{-1})^\eta = (Y^{-1})^\eta$.

**Lemma 2.9.** Let $X$ and $Y$ be permutation operators. Then $XY = YX$ if and only if the associated permutations commute and $(XY)^\eta = (YX)^\eta$ for every $\eta \in \hat{G}$.

To determine whether $J$, $I_a$, or $J_a$ commutes with $T(p)$ we will apply the preceding lemmas. According to Theorem 1.9 and (2.4), the conditions (a) $I_a T(p) = T(p) I_a$, (b) $J_a T(p) = T(p) J_a$, and (c) $J T(p) = T(p) J$ are equivalent to

(a) $S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 = B_3(\chi) T_3 K S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3$,  

(b) $S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 B_3(\chi) T_3 K = B_3(\chi) T_3 K S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3$,  

(c) $S_3^{-1} K S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 B_3(\chi) T_3 K = B_3(\chi) T_3 K S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3$.

By manipulation of these equations and the fact that operators with different subscripts commute, it is verified that (2.10) is equivalent to

(a) $S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 B_3(\chi) T_3 K = B_3(\chi) T_3 K S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3$,

(b) $S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 B_3(\chi) T_3 K = B_3(\chi) T_3 K S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3$,

(c) $S_3^{-1} K S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3 B_3(\chi) T_3 K = B_3(\chi) T_3 K S_3^{-1} M_3 \overline{F}_3 \overline{S}_3 \overline{G}_3 \overline{F}_3$.

This set of conditions is an improvement in that $(S_1^{-1})^2 f(x, y, t) = f(-x, y, t)$ and $(S_2^{-1})^2 f(x, y, t) = f(x, -y, t)$ are particularly simple operators.

By taking the $\eta$-transform of each side of the equations in (2.11), we can use Lemma 2.9 to determine whether these equations hold. First, we need the $\eta$-transforms of all the operators which occur in (2.11). For $x \in \mathbb{V}$ let $|x| = x_1 + x_2 + \cdots + x_m$.

**Theorem 2.12.** Fix $\eta \in \hat{G}$. The following are $\eta$-transforms:

(a) $((S_1^{-1})^2)^\eta f(x, y, t) = \eta(-I, I, 1)f(x, y, t)$,

(b) $((S_2^{-1})^2)^\eta f(x, y, t) = \eta(I, -I, 1)f(x, y, t)$,

(c) $K((S_2^{-1})^2)^\eta f(x, y, t) = \eta(I, -I, 1)f(y, x, t - |x| - |y|)$,

(d) $(K^{-1})^\eta f(x, y, t) = \eta(-I, I, 1)f(y, x, t + |x| + |y|)$,

(e) $B_3(\chi)^\eta f(x, y, t) = f(x, y, t - \delta)$,

(f) $T_3^2 f(x, y, t) = \eta(I, I, -1)f(x, y, -t)$,

(g) $M_3^2 f(x, y, t) = f(x, y, t)$,

(h) $\overline{F}_3^2 f(x, y, t) = w_\eta(t)f(x, y, -t)$, and
(i) \((\mathcal{F}_3^{-1})^7 f(x, y, t) = \eta(I, I, -1)w_\eta(t)f(x, y, -t)\) where
\[
w_\eta(t) = \begin{cases} 
2^\mu \Gamma\left(\frac{it}{2} + \frac{1}{4}\right) / \Gamma\left(-\frac{it}{2} + \frac{1}{4}\right) & \text{if } \eta(I, I, -1) = 1, \\
i2^\mu \Gamma\left(\frac{it}{2} + \frac{3}{4}\right) / \Gamma\left(-\frac{it}{2} + \frac{3}{4}\right) & \text{if } \eta(I, I, -1) = -1.
\end{cases}
\]

**Proof.** The computations are straightforward except in the case of (h). But one can show that
\[
\mathcal{F}_3^7 f(x, y, t) = \begin{cases} 
f(x, y, -t)(2\pi)^{-1/2} \int_0^\infty 2(\cos \lambda)\lambda^{\sigma - 1/2} d\lambda & \text{if } \eta(I, I, -1) = 1, \\
f(x, y, -t)(2\pi)^{-1/2} \int_0^\infty 2i(\sin \lambda)\lambda^{\sigma - 1/2} d\lambda & \text{if } \eta(I, I, -1) = -1.
\end{cases}
\]
These integrals are the ordinary Mellin transforms of the function \(2 \cos t\) and \(2i \sin t\) evaluated at \(it + 1/2\), which are found in [4, p. 196] and, after simplification, yield formula (h).

Fix \(\eta \in \hat{G}\) and let \(e, \mu \in \hat{G}\) be defined as before by \(e(e_1, e_2, e_3) = e_3\) and \(\mu(e_1, e_2, e_3) = \eta(e_3e_2, e_3e_1, e_3)\). From Theorem 2.12 and Lemma 2.8, one obtains the following \(\eta\)-transforms of the expressions in (2.11): From (2.11)(a),
\[
(\mathcal{F}_3^{-1}M_3\mathcal{F}_3B_3(\chi) T_3K)^7 f(x, y, t)
\]
(2.13)
\[
= -\eta(I, -I, 1)w_\eta(t)w_\mu(-t)f(y, x, \sigma - t - |x| - |y|)
\]
and
\[
(\mathcal{F}_3^{-1}M_3B_3(\chi) T_3K)^7 f(x, y, t) = \eta(-I, I, 1)w_\eta(\sigma - t - |x| - |y|)
\]
(2.14)
\[
\cdot w_\mu(t + |x| + |y| - \sigma)f(y, x, \sigma - t - |x| - |y|).
\]
From (2.11)(b),
\[
(\mathcal{F}_3^{-1}M_3K^{-1}\mathcal{F}_3B_3(\chi) T_3)^7 f(x, y, t) = -\eta(I, -I, 1)w_\eta(t + \sigma)
\]
(2.15)
\[
w_\mu(-t - \sigma + |x| + |y|)f(y, x, \sigma - t + |x| + |y|)
\]
and
\[
(T_3K^{-1}(\mathcal{F}_1^{-1})^2(\mathcal{F}_2^{-1})^2\mathcal{F}_3^{-1}M_3K^{-1}\mathcal{F}_3K^{-1})^7 f(x, y, t)
\]
(2.16)
\[
= \eta(I, -I, 1)w_\eta(-t + |x| + |y|)w_\mu(t)f(y, x, \sigma - t + |x| + |y|).
\]
From (2.11)(c)
\[
(\mathcal{F}_3^{-1}K^{-1}\mathcal{F}_3B_3(\chi) T_3)^7 f(x, y, t)
\]
(2.17)
\[
= \eta(I, -I, 1)w_\eta(t + \sigma)w_\mu(-t - \sigma + |x| + |y|)f(y, x, \sigma - t + |x| + |y|)
\]
and
\[
(T_3K^{-1}(\mathcal{F}_1^{-1})^2(\mathcal{F}_2^{-1})^2\mathcal{F}_3^{-1}K^{-1}\mathcal{F}_3K^{-1})^7 f(x, y, t)
\]
(2.18)
\[
= \eta(I, -I, 1)w_\mu(t)w_\eta(-t + |x| + |y|)f(y, x, \sigma - t + |x| + |y|).
Applying Lemma 2.9 and the formulas (2.13) and (2.14), we find that (2.11)(a) holds, and hence \( I_a \) commutes with \( T(p) \), if and only if
\[
\eta(I, -I, 1)w_{en}(t)w_{\mu}(-t) = \eta(-I, I, 1)w_{en}(\sigma - t - |x| - |y|)w_{\mu}(t + |x| + |y| - \sigma)
\]  
(2.19)
for all \( \eta \in \hat{G} \) and almost every \((x, y, t) \in Z\). In fact, since the two sides of (2.19) are continuous functions on \( Z \), this equation must hold for every \((x, y, t) \in Z\). In particular, let \( \eta = \theta \) be the trivial character, \( t = 0 \), and \( |x| + |y| = \sigma \); then (2.19) becomes \(-i = -w_e(0)w_p(0) = w_e(0)w_p(0) = i\), which is impossible. Thus \( I_a \) does not commute with \( T(p) \) regardless of the choice of the unitary character \( \chi \) of \( H \).

Similarly, (2.11)(b) holds, and hence \( J_a \) commutes with \( T(p) \), if and only if
\[
\eta(I, -I, 1)w_{en}(t + \sigma)w_{\mu}(-t - \sigma + |x| + |y|) = \eta(-I, I, 1)w_{en}(t - |x| + |y|)
\]  
(2.20)
for every \((x, y, t) \in Z\) and every \( \eta \in \hat{G} \). In particular, with \( \eta = \theta \), the trivial character, \( t = 0 \), and \( |x| + |y| = \sigma \), (2.20) becomes \(-w_e(\sigma) = -w_e(\sigma)w_p(0) = w_e(\sigma)w_p(0) = w_e(\sigma)\) which is impossible since \( w_e(\sigma) \) is never zero. Thus, \( J_a \) does not commute with \( T(p) \) regardless of the choice of \( \chi \).

Finally, (2.11)(c) holds, and hence \( J \) commutes with \( T(p) \), if and only if
\[
\eta(I, -I, 1)w_{en}(t + \sigma)w_{\mu}(-t - \sigma + |x| + |y|) = w_{\mu}(t)w_{en}(-t + |x| + |y|)
\]  
(2.21)
for every \((x, y, t) \in Z\) and every \( \eta \in \hat{G} \). If \( \sigma = 0 \) then (2.21) obviously holds. Thus, \( J \) commutes with \( T(p) \) in case \( \chi \) is the trivial character of \( H \). We will now show that the assumption of (2.21) implies that \( \sigma = 0 \), so that \( J \) does not commute with \( T(p) \) in any other case.

Let \( \eta \) and hence \( \mu \), be the trivial character of \( G \). Let \( \gamma(t) = \Gamma(it/2 + 1/4) \); then (2.21) implies
\[
\frac{\gamma(t + \sigma)\gamma(-t - \sigma + |x| + |y|)}{\gamma(-t - \sigma)\gamma(t + \sigma - |x| - |y|)} = \frac{\gamma(-t + |x| + |y|)}{\gamma(t - |x| - |y|)}\gamma(t)
\]  
(2.22)
for every \((x, y, t) \in Z\). Upon rearrangement, this becomes
\[
\frac{\gamma(t + \sigma)\gamma(-t)}{\gamma(t)\gamma(-t - \sigma)} = \frac{\gamma(t + \sigma - |x| - |y|)}{\gamma(t - |x| - |y|)}\gamma(-t - \sigma + |x| + |y|),
\]  
(2.23)
and uniqueness of Haar measure on \((R, +)\) shows that
\[
\frac{\gamma(t + \sigma)}{\gamma(t)} \cdot \frac{\gamma(-t)}{\gamma(-t - \sigma)} = \frac{\gamma(\sigma)}{\gamma(-\sigma)} \quad \text{(a constant)}.
\]  
(2.24)
Differentiation of this equation with respect to \( t \) at \( t = 0 \) results in
\[
\gamma'(\sigma)/\gamma(\sigma) + \gamma'(-\sigma)/\gamma(-\sigma) = 2\gamma'(0)/\gamma(0).
\]  
(2.25)
In terms of the Gamma function, since \( \gamma(\sigma) = i\Gamma(i\sigma/2 + 1/4)/2 \), this becomes
\[
\frac{\Gamma'(i\sigma/2 + 1/4)}{\Gamma(i\sigma/2 + 1/4)} + \frac{\Gamma'(-i\sigma/2 + 1/4)}{\Gamma(-i\sigma/2 + 1/4)} = 2\frac{\Gamma'(1/4)}{\Gamma(1/4)}.
\]  
(2.26)
Let \( \psi(z) = \Gamma'(z)/\Gamma(z) \), and set \( z = i\sigma/2 + 1/4 \). Then (2.26) can be rewritten as
\[
\text{Re } \psi(z) = \psi(\text{Re } z).
\]  
(2.27)
The next lemma, which follows from a well-known expansion for $\psi$ [1, p. 189], finishes the proof.

**Lemma 2.28.** Let $\psi(z) = \Gamma'(z)/\Gamma(z)$, with $z \in \mathbb{C}$ and $\text{Re } z > 0$. Then $\text{Re } \psi(z) = \psi(\text{Re } z)$ if and only if $\text{Im } z = 0$.

We have proved that $J$ commutes with $T(p)$ if and only if $\chi$ is the trivial character.

We give a formal statement of the results we have just proved. To this end recall that $T = T_\chi$, and let $P_\pm = (I \pm J)/2$ and $L^2(Z)^\pm = P_\pm(L^2(Z))$.

**Theorem 2.29.** Let $\chi$ be a unitary character of $\overline{H}$ and $T$ be the degenerate principal series representation of $\text{Sp}(n, \mathbb{R})$ induced by $\chi$ and acting in $L^2(Z)$. If $\chi$ is not the trivial character then $T$ is irreducible. If $\chi$ is the trivial character then $J \in \mathcal{O}(T)$, the subrepresentations $T^\pm$ of $T$ on $L^2(Z)^\pm$ are irreducible, and $T = T^+ \oplus T^-$. 

**References**


Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056