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LIMIT SETS OF AUTOMORPHISM GROUPS OF A TREE

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ABSTRACT. Let X be an infinite and locally finite tree and G be an arbitrary automorphism group of X. The limit set $L(G)$ of G in the boundary B of X is studied.

1. Introduction. A tree is a nonempty connected graph X without circuit. One can associate a tree to SL_2 over the field Q_p as a special case of the Bruhat-Tits building which is the p-adic analogue of homogeneous symmetric spaces of real semisimple Lie groups [1], [2], [7], [8]. Other applications are the graph $\Gamma(G, P)$ of a group G and a subset P of G and algebraic geometry [7]. Automorphism groups and harmonic analysis on a tree have been investigated by Tits [8] and Cartier [1], [2] respectively.

In the theory of hyperbolic plane and fuchsian groups one has the definition of the limit set $L(G)$ of a subgroup $G \subseteq SL_2(R)$. This concept will be defined for an arbitray automorphism group G on an infinite and locally finite tree X. We shall obtain some general facts about $L(G)$ such as the cardinality of $L(G)$ and the relative size of $L(G)$ in the boundary B of X. In [5] and [6], Kulkarni has considered this question for a general class of topological spaces. He has also investigated the connection with the theory of ends of Freudenthal and Hopf. Our main result is a theorem similar to that of Ahlfors for Kleinian groups [3].

2. Tree. Let X be an infinite and locally finite tree with vertex set S and edge set A. Let s and t be vertices and be connected. Then the edge $\{s, t\}$ is in A. (For simplicity we do not emphasize orientation here.) A path in X is a sequence $C = [s_0, s_1, \ldots, s_n]$ of vertices such that s_{i-1} is connected to s_i for $1 < i < n$. The length of C is n. If all the vertices in a path C_0 are distinct, then this unique path C_0 is called a geodesic. The length of the geodesic joining vertices s to t is called the distance $d(s, t)$ between them. Let s be a vertex. The infinite path originated from s is an infinite sequence $C = [s_0, s_1, \ldots, s_n \ldots]$ of vertices such that $s_0 = s$ and s_n and s_{n+1} are connected for each $n > 0$. C is called an infinite geodesic if $s_i \neq s_j$ for $i \neq j$. The relation that two infinite geodesics C and C' have infinitely many common vertices is an equivalence relation. Each equivalence class is called an end. The set B of all ends is the boundary to X. We can join each s in S and each b in B by an infinite geodesic. Moreover, each pair $b \neq b'$ in B can be joined by an infinite geodesic. With the topology described in Theorem 1.1 of [2], the space

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$\hat{S} = S \cup B$ is a compact and totally disconnected space such that S, with the discrete topology, is an open dense subspace of $\hat{S}$. Let s and s' be two vertices. If the vertex t moves along the infinite geodesic representing the end b, then $d(s, t) - d(s', t)$ is a finite constant $\delta_b(s, s')$. A horocycle H through s is the set of vertices s' such that $\delta_b(s, s') = 0$. There is a partition of S into a disjoint union of horocycles $\{H_n\}_{n=0}^{\infty}$.

Let g be an automorphism of X. Tits [8] has proved that one of the following holds:

(1) there exists s in S such that $g(s) = s$,

(2) there exist s and s' in S such that $\{s, s'\}$ is an edge and $g(s) = s'$ and $g(s') = s$,

(3) there exist $\{s_n\}_{n=-\infty}^{\infty}$ such that $\{s_n\ s_{n+1}\}$ is an edge and $g(s_n) = s_{n+k}$ for any n and some $k > 0$.

Furthermore, he has proved that if G is an automorphism group of X and G has no element of type (3), then G leaves a vertex s, an edge $\{s, s'\}$ or an end b (together with all horocycles associated to b) invariant. It is obvious that automorphisms of X are isometries with respect to the distance $d(s, s')$ defined above. Each automorphism g of X can be extended to act on the boundary B. The action is a homeomorphism.

3. Limit set. With the topology on $\hat{S} = S \cup B$, we define the limit set $L(G)$ of an automorphism group G to be the set of accumulation points in B of an orbit Gs of s. That is $L(G) = \overline{Gs} \cap B$. There is the question as to whether $L(G)$ is independent of the choice of the vertex s. Let s and s' be two vertices with finite distance $d(s, s') = d$. We have two orbits Gs and Gs'. Each b in $L(G)$ is the limit $\lim g_n s$ for some sequence g_n in G. Let b' be the limit $\lim g_n s'$. Since $d(s, s') = d(g_n s, g_n s') = d$ and $d(b, b') = \infty$, we have $b = b'$. Consequently, $L(G)$ is independent of the choice of s.

PROPOSITION 1. (i) *$L(G)$ is a closed invariant subset of B under G.*

(ii) *Let A be any closed invariant subset of B under G and let A contain more than one point. Then $A \supset L(G)$ and $L(G)$ is minimal under the action of G on B.*

(iii) *The set of fixed points of elements of type* (3) (*if they exist*) *is dense in $L(G)$.*

PROOF. It is clear that $L(G)$ is closed. The invariance of $L(G)$ is seen by $b = \lim g\, g_n(s) = g \lim g_n(s) = g(b)$. To prove (ii), let b and b' be in A. Let b'' be in $L(G)$ such that $\lim g_n s = b''$. The two infinite points b and b' can be joined by an infinite geodesic. We choose s to be on that geodesic. Then either $g_n b$ or $g_n b'$ must converge to b''. Thus $A \supset L(G)$. (iii) follows from (ii) directly. In fact, each element of type (3) has two fixed points in B. The closure of fixed points of elements of type (3) in G can be taken as A.

PROPOSITION 2. *The cardinality of $L(G)$ of an automorphism group G is* 0, 1, 2 *or* ∞.

PROOF. Either G contains no elements of type (3) or it does contain them. In the first case, G has a fixed point s or an invariant edge $\{s, s'\}$ or a fixed end b. If G

has a fixed point s or an invariant edge $\{s, s'\}$, then $L(G)$ is empty. If G has a fixed end b, then G leaves horocycles associated to b invariant.

The orbit Gs will converge to b, if we take s to be a vertex on a horocycle associated to b. The limit set $L(G)$ consists of one point. In the second case, $L(G)$ contains at least two points, because two ends b and b' of the infinite geodesic translated by an element g of type (3) will be limit points. If $L(G)$ has more than two points, then $L(G)$ has infinitely many points. In fact, if $L(G)$ has finitely many points, then the nth power g^n of any element g of G will leave every point in $L(G)$ fixed when n is sufficiently large. Then g^n has at least two fixed ends and must translate the infinite geodesic joining them. Thus g is of type (3) and $L(G)$ can have only two points.

REMARK. We may obtain a classification (see Corollary 3.5 of **[8]** also) of automorphism groups on an infinite and locally finite tree X. There are the following possibilities:

(i) G has a common fixed point s in S or has a common invariant edge $\{s, t\}$ in A. $L(G)$ is empty.

(ii) G has a common fixed end b in B and $L(G)$ consists of one point.

(iii) G has two common fixed ends b_1 and b_2 in B and $L(G)$ consists of two points.

(iv) G has an invariant proper subtree X' (not a vertex, an edge or an infinite geodesic) of X and $L(G)$ is the boundary B' of X'.

(v) G acts minimally on B and $L(G) = B$.

From the above, we also obtain a classification of quotient graphs $G \setminus X$ of X if G acts freely on X. There are the following possibilities:

(i) $G \setminus X$ is parabolic if $L(G)$ has one point and $G \setminus X$ is a union of graphs $G \setminus H_n$, which are the quotient graphs of horocycles H_n by G.

(ii) $G \setminus X$ is axial if $L(G)$ consists of two points. There is an infinite geodesic C which is translated by G and $G \setminus C$ is the unique circuit such that the injection $G \setminus C \to G \setminus X$ is a homotopy equivalence **[7]**.

(iii) $G \setminus X$ is fuchsian if $L(G)$ consists of infinitely many points. This is the most interesting case and offers a great variety of quotient graphs.

4. Boundary measure. We use §§2.5, 3.3 and 3.4 of **[2]** to investigate the boundary measure of $L(G)$ of an automorphism group G. We refer to **[1]**, **[2]** for details. Let X be an infinite and locally finite tree. For each finite path a, we assign a positive number $p(a)$ such that $p(aa') = p(a)p(a')$ if the composite path aa' is defined, and for $s, s' \in S$, $\sum p(ss') = 1$ for all s' connected to s. If h is a function on S, we define the function Ph on S by $Ph(s) = \sum p(ss')h(s')$ over the set of all vertices s' connected to s. Let $\Delta h = Ph - h$ be the Laplace operator and let $K(s, b)$ be the Martin kernel of X. There exists a positive measure μ of mass 1 on B such that

$$\int_B K(s, b)\mu(db) = 1$$

for each s in S. This measure is called the Poisson measure. Consider a random walk $\{X_0, X_1, \ldots, X_n, \ldots\}$ on S in the following manner. If $X_0 = 0$ is the original point and $X_n = s$, then the permissible values of X_{n+1} are vertices s' connected to s

with probability $p(s, s')$ from s to s'. By the Borel-Cantelli lemma, X_n converges to a point X_∞ in B with probability 1, with respect to the probability law given by the Poisson measure μ on B. If h is a bounded harmonic function, the Martingale theorem shows that the sequence of random variables $\{h(X_n)\}$ converges to a random variable Y with probability 1. There exists a bounded Borel function ϕ on B such that $Y = \phi(X_\infty)$ and one has the Poisson integral formula

$$h(s) = \int_B K(s, b)\phi(b)\mu(db), \quad \text{any } s \in S.$$

The radial convergence theorem of Fatou holds in the following sense: $\phi(b) = \lim h(s_n)$, almost everywhere in B with respect to μ, on an infinite geodesic $[s_0, s_1, \ldots, s_n, \ldots]$ joining the original point 0 to b.

A limit point b in $L(G)$ is said to be an approximation limit point if for any sequence $\{s_n\}$ in a geodesic approaching to b, there exists a sequence $\{g_n\}$ in G such that $g_n(s_n)$ lies in a finite set (compact set). This notion is originally due to Hedlund ([3], [6] for further references). Because $g_n(s_n)$ lies in a finite set, without loss of generality we may assume that they are all equal. Thus, we can find a sequence $\{s_k\}$ converging to b and $\{g_k\} \subseteq G$ such that $\{g_k(s_k)\}$ is a point in S. It is obvious that fixed points of elements of type (3) are approximation limit points. Also the set $A(G)$ of approximation limit points of G is G-invariant.

THEOREM. *Let X be an infinite and locally finite tree and G be an automorphism group of X. Then the Poisson measure $\mu(A(G))$ of the set $A(G)$ of approximation limit points of G is either* 0 *or* 1. *If $L(G)$ contains almost all approximation limit points, then the Poisson measure $\mu(L(G))$ is either* 0 *or* 1.

PROOF. The characteristic function ϕ on B of $A(G)$ is G-invariant and takes values 0 and 1. Let h be the harmonic function

$$h(s) = \int_B K(s, b)\phi(b)\mu(db).$$

Then h is also G-invariant. If the measure $\mu(A(G))$ is positive, then there is a point b in $A(G)$ such that Fatou's convergence theorem can be applied. There exists an infinite geodesic C approaching b such that the radial limit of $h(s)$ along C approaching b must exist and is equal to 1. Note that $0 < h(s) < 1$ for all s in S. Since each b in $A(G)$ is an approximation limit point, there is a sequence $\{s_n\}$ on C converging to b and there is a sequence $\{g_n\}$ in G such that $g_n(s_n)$ is a point s_0 in S. Then the G-invariance and continuity implies that $h(s_0) = 1$. But 1 is the upper bound for $h(s)$. Hence $h(s)$ attains its maximum at an interior point s_0 in S. $h(s)$ has to be the constant 1 and

$$\begin{aligned}
\mu(A(G)) &= \int_{A(G)} K(s, b)\mu(db) \\
&= \int_B K(s, b)\phi(b)\mu(db) \qquad (\phi \text{ is the characteristic function}) \\
&= \int_B K(s, b)\mu(db) \qquad \left(\text{since } \phi(b) = \lim_{s \to b} h(s) = 1\right) = 1.
\end{aligned}$$

If $L(G)$ contains almost all approximation limit points (that is, except a set of measure zero), then we have $\mu(L(G))$ is 0 to 1 by the above result of $A(G)$.

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