LIMIT SETS OF AUTOMORPHISM GROUPS OF A TREE

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Abstract. Let $X$ be an infinite and locally finite tree and $G$ be an arbitrary automorphism group of $X$. The limit set $L(G)$ of $G$ in the boundary $B$ of $X$ is studied.

1. Introduction. A tree is a nonempty connected graph $X$ without circuit. One can associate a tree to $SL_2$ over the field $Q_p$ as a special case of the Bruhat-Tits building which is the $p$-adic analogue of homogeneous symmetric spaces of real semisimple Lie groups [1], [2], [7], [8]. Other applications are the graph $\Gamma(G, P)$ of a group $G$ and a subset $P$ of $G$ and algebraic geometry [7]. Automorphism groups and harmonic analysis on a tree have been investigated by Tits [8] and Cartier [1], [2] respectively.

In the theory of hyperbolic plane and fuchsian groups one has the definition of the limit set $L(G)$ of a subgroup $G \subseteq SL_2(R)$. This concept will be defined for an arbitrary automorphism group $G$ on an infinite and locally finite tree $X$. We shall obtain some general facts about $L(G)$ such as the cardinality of $L(G)$ and the relative size of $L(G)$ in the boundary $B$ of $X$. In [5] and [6], Kulkarni has considered this question for a general class of topological spaces. He has also investigated the connection with the theory of ends of Freudenthal and Hopf. Our main result is a theorem similar to that of Ahlfors for Kleinian groups [3].

2. Tree. Let $X$ be an infinite and locally finite tree with vertex set $S$ and edge set $A$. Let $s$ and $t$ be vertices and be connected. Then the edge $\{s, t\}$ is in $A$. (For simplicity we do not emphasize orientation here.) A path in $X$ is a sequence $C = [s_0, s_1, \ldots, s_n]$ of vertices such that $s_{i-1}$ is connected to $s_i$ for $1 < i < n$. The length of $C$ is $n$. If all the vertices in a path $C_0$ are distinct, then this unique path $C_0$ is called a geodesic. The length of the geodesic joining vertices $s$ to $t$ is called the distance $d(s, t)$ between them. Let $s$ be a vertex. The infinite path originated from $s$ is an infinite sequence $C = [s_0, s_1, \ldots, s_n, \ldots]$ of vertices such that $s_0 = s$ and $s_n$ and $s_{n+1}$ are connected for each $n > 0$. $C$ is called an infinite geodesic if $s_i \neq s_j$ for $i \neq j$. The relation that two infinite geodesics $C$ and $C'$ have infinitely many common vertices is an equivalence relation. Each equivalence class is called an end. The set $B$ of all ends is the boundary to $X$. We can join each $s$ in $S$ and each $b$ in $B$ by an infinite geodesic. Moreover, each pair $b \neq b'$ in $B$ can be joined by an infinite geodesic. With the topology described in Theorem 1.1 of [2], the space
\[
\hat{S} = S \cup B
\]
is a compact and totally disconnected space such that \(S\), with the discrete topology, is an open dense subspace of \(\hat{S}\). Let \(s\) and \(s'\) be two vertices. If the vertex \(t\) moves along the infinite geodesic representing the end \(b\), then \(d(s, t) - d(s', t)\) is a finite constant \(\delta_b(s, s')\). A horocycle \(H\) through \(s\) is the set of vertices \(s'\) such that \(\delta_b(s, s') = 0\). There is a partition of \(S\) into a disjoint union of horocycles \(\{H_n\}_{n=0}^{\infty}\).

Let \(g\) be an automorphism of \(X\). Tits \([8]\) has proved that one of the following holds:

1. there exists \(s\) in \(S\) such that \(g(s) = s\),
2. there exist \(s\) and \(s'\) in \(S\) such that \(\{s, s'\}\) is an edge and \(g(s) = s'\) and \(g(s') = s\),
3. there exist \(\{s_n\}\) such that \(\{s_n, s_{n+1}\}\) is an edge and \(g(s_n) = s_{n+k}\) for any \(n\) and some \(k > 0\).

Furthermore, he has proved that if \(G\) is an automorphism group of \(X\) and \(G\) has no element of type (3), then \(G\) leaves a vertex \(s\), an edge \(\{s, s'\}\) or an end \(b\) (together with all horocycles associated to \(b\)) invariant. It is obvious that automorphisms of \(X\) are isometries with respect to the distance \(d(s, s')\) defined above. Each automorphism \(g\) of \(X\) can be extended to act on the boundary \(B\). The action is a homeomorphism.

### 3. Limit set

With the topology on \(\hat{S} = S \cup B\), we define the limit set \(L(G)\) of an automorphism group \(G\) to be the set of accumulation points in \(B\) of an orbit \(Gs\) of \(s\). That is \(L(G) = \overline{Gs} \cap B\). There is the question as to whether \(L(G)\) is independent of the choice of the vertex \(s\). Let \(s\) and \(s'\) be two vertices with finite distance \(d(s, s') = d\). We have two orbits \(Gs\) and \(G_{s'}\). Each \(b\) in \(L(G)\) is the limit \(g_n s\) for some sequence \(g_n\) in \(G\). Let \(b'\) be the limit \(\lim g_n s'\). Since \(d(s, s') = d(g_n s, g_n s') = d\) and \(d(b, b') = \infty\), we have \(b = b'\). Consequently, \(L(G)\) is independent of the choice of \(s\).

**Proposition 1.** (i) \(L(G)\) is a closed invariant subset of \(B\) under \(G\).

(ii) Let \(A\) be any closed invariant subset of \(B\) under \(G\) and let \(A\) contain more than one point. Then \(A \supset L(G)\) and \(L(G)\) is minimal under the action of \(G\) on \(B\).

(iii) The set of fixed points of elements of type (3) (if they exist) is dense in \(L(G)\).

**Proof.** It is clear that \(L(G)\) is closed. The invariance of \(L(G)\) is seen by \(b = \lim g g_n(s) = g \lim g_n(s) = g(b)\). To prove (ii), let \(b\) and \(b'\) be in \(A\). Let \(b''\) be in \(L(G)\) such that \(\lim g_n s = b''\). The two infinite points \(b\) and \(b'\) can be joined by an infinite geodesic. We choose \(s\) to be on that geodesic. Then either \(g_n b\) or \(g_n b'\) must converge to \(b''\). Thus \(A \supset L(G)\). (iii) follows from (ii) directly. In fact, each element of type (3) has two fixed points in \(B\). The closure of fixed points of elements of type (3) in \(G\) can be taken as \(A\).

**Proposition 2.** The cardinality of \(L(G)\) of an automorphism group \(G\) is 0, 1, 2 or \(\infty\).

**Proof.** Either \(G\) contains no elements of type (3) or it does contain them. In the first case, \(G\) has a fixed point \(s\) or an invariant edge \(\{s, s'\}\) or a fixed end \(b\). If \(G\)
has a fixed point \( s \) or an invariant edge \( \{s, s'\} \), then \( L(G) \) is empty. If \( G \) has a fixed end \( b \), then \( G \) leaves horocycles associated to \( b \) invariant.

The orbit \( Gs \) will converge to \( b \), if we take \( s \) to be a vertex on a horocycle associated to \( b \). The limit set \( L(G) \) consists of one point. In the second case, \( L(G) \) contains at least two points, because two ends \( b \) and \( b' \) of the infinite geodesic translated by an element \( g \) of type (3) will be limit points. If \( L(G) \) has more than two points, then \( L(G) \) has infinitely many points. In fact, if \( L(G) \) has finitely many points, then the \( n \)th power \( g^n \) of any element \( g \) of \( G \) will leave every point in \( L(G) \) fixed when \( n \) is sufficiently large. Then \( g^n \) has at least two fixed ends and must translate the infinite geodesic joining them. Thus \( g \) is of type (3) and \( L(G) \) can have only two points.

**Remark.** We may obtain a classification (see Corollary 3.5 of [8] also) of automorphism groups on an infinite and locally finite tree \( X \). There are the following possibilities:

(i) \( G \) has a common fixed point \( s \) in \( S \) or has a common invariant edge \( \{s, t\} \) in \( A \). \( L(G) \) is empty.

(ii) \( G \) has a common fixed end \( b \) in \( B \) and \( L(G) \) consists of one point.

(iii) \( G \) has two common fixed ends \( b_1 \) and \( b_2 \) in \( B \) and \( L(G) \) consists of two points.

(iv) \( G \) has an invariant proper subtree \( X' \) (not a vertex, an edge or an infinite geodesic) of \( X \) and \( L(G) \) is the boundary \( B' \) of \( X' \).

(v) \( G \) acts minimally on \( B \) and \( L(G) = B \).

From the above, we also obtain a classification of quotient graphs \( G \setminus X \) of \( X \) if \( G \) acts freely on \( X \). There are the following possibilities:

(i) \( G \setminus X \) is parabolic if \( L(G) \) has one point and \( G \setminus X \) is a union of graphs \( G \setminus H_n \), which are the quotient graphs of horocycles \( H_n \) by \( G \).

(ii) \( G \setminus X \) is axial if \( L(G) \) consists of two points. There is an infinite geodesic \( C \) which is translated by \( G \) and \( G \setminus C \) is the unique circuit such that the injection \( G \setminus C \to G \setminus X \) is a homotopy equivalence [7].

(iii) \( G \setminus X \) is fuchsian if \( L(G) \) consists of infinitely many points. This is the most interesting case and offers a great variety of quotient graphs.

### 4. Boundary measure

We use §§2.5, 3.3 and 3.4 of [2] to investigate the boundary measure of \( L(G) \) of an automorphism group \( G \). We refer to [1], [2] for details. Let \( X \) be an infinite and locally finite tree. For each finite path \( a \), we assign a positive number \( p(a) \) such that \( p(aa') = p(a)p(a') \) if the composite path \( aa' \) is defined, and for \( s, s' \in S \), \( \sum p(ss') = 1 \) for all \( s' \) connected to \( s \). If \( h \) is a function on \( S \), we define the function \( Ph \) on \( S \) by \( Ph(s) = \sum p(ss')h(s') \) over the set of all vertices \( s' \) connected to \( s \). Let \( \Delta h = Ph - h \) be the Laplace operator and let \( K(s, b) \) be the Martin kernel of \( X \). There exists a positive measure \( \mu \) of mass 1 on \( B \) such that

\[
\int_B K(s, b) \mu(db) = 1
\]

for each \( s \) in \( S \). This measure is called the Poisson measure. Consider a random walk \( \{X_0, X_1, \ldots, X_n, \ldots\} \) on \( S \) in the following manner. If \( X_0 = 0 \) is the original point and \( X_n = s \), then the permissible values of \( X_{n+1} \) are vertices \( s' \) connected to \( s \).
with probability $p(s, s')$ from $s$ to $s'$. By the Borel-Cantelli lemma, $X_n$ converges to a point $X_\infty$ in $B$ with probability 1, with respect to the probability law given by the Poisson measure $\mu$ on $B$. If $h$ is a bounded harmonic function, the Martingale theorem shows that the sequence of random variables $\{h(X_n)\}$ converges to a random variable $Y$ with probability 1. There exists a bounded Borel function $\phi$ on $B$ such that $Y = \phi(X_\infty)$ and one has the Poisson integral formula

$$h(s) = \int_B K(s, b)\phi(b)\mu(db), \quad \text{any } s \in S.$$ 

The radial convergence theorem of Fatou holds in the following sense: $\phi(b) = \lim h(s_n)$, almost everywhere in $B$ with respect to $\mu$, on an infinite geodesic $[s_0, s_1, \ldots, s_n, \ldots]$ joining the original point 0 to $b$.

A limit point $b$ in $L(G)$ is said to be an approximation limit point if for any sequence $\{s_n\}$ in a geodesic approaching to $b$, there exists a sequence $\{g_n\}$ in $G$ such that $g_n(s_n)$ lies in a finite set (compact set). This notion is originally due to Hedlund ([3], [6] for further references). Because $g_n(s_n)$ lies in a finite set, without loss of generality we may assume that they are all equal. Thus, we can find a sequence $\{s_k\}$ converging to $b$ and $\{g_k\} \subseteq G$ such that $\{g_k(s_k)\}$ is a point in $S$. It is obvious that fixed points of elements of type (3) are approximation limit points. Also the set $A(G)$ of approximation limit points of $G$ is $G$-invariant.

**Theorem.** Let $X$ be an infinite and locally finite tree and $G$ be an automorphism group of $X$. Then the Poisson measure $\mu(A(G))$ of the set $A(G)$ of approximation limit points of $G$ is either 0 or 1. If $L(G)$ contains almost all approximation limit points, then the Poisson measure $\mu(L(G))$ is either 0 or 1.

**Proof.** The characteristic function $\phi$ on $B$ of $A(G)$ is $G$-invariant and takes values 0 and 1. Let $h$ be the harmonic function

$$h(s) = \int_B K(s, b)\phi(b)\mu(db).$$

Then $h$ is also $G$-invariant. If the measure $\mu(A(G))$ is positive, then there is a point $b$ in $A(G)$ such that Fatou's convergence theorem can be applied. There exists an infinite geodesic $C$ approaching $b$ such that the radial limit of $h(s)$ along $C$ approaching $b$ must exist and is equal to 1. Note that $0 < h(s) < 1$ for all $s$ in $S$. Since each $b$ in $A(G)$ is an approximation limit point, there is a sequence $\{s_n\}$ on $C$ converging to $b$ and there is a sequence $\{g_n\}$ in $G$ such that $g_n(s_n)$ is a point $s_0$ in $S$. Then the $G$-invariance and continuity implies that $h(s_0) = 1$. But 1 is the upper bound for $h(s)$. Hence $h(s)$ attains its maximum at an interior point $s_0$ in $S$. $h(s)$ has to be the constant 1 and

$$\mu(A(G)) = \int_{A(G)} K(s, b)\mu(db) = \int_B K(s, b)\phi(b)\mu(db) \quad (\phi \text{ is the characteristic function})$$

$$= \int_B K(s, b)\mu(db) \quad (\text{since } \phi(b) = \lim_{s \to b} h(s) = 1) = 1.$$
If $L(G)$ contains almost all approximation limit points (that is, except a set of measure zero), then we have $\mu(L(G))$ is 0 to 1 by the above result of $A(G)$.

REFERENCES

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