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NONEXTENDED QUADRATIC FORMS OVER POLYNOMIAL RINGS OVER POWER SERIES RINGS

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ABSTRACT. If R is a complete discrete valuation ring, then every quadratic space over $R[T]$ is extended from R. We here show by an example that a corresponding result for higher-dimensional complete regular local rings is not valid.

It was proved in [3] that if R is any complete discrete valuation ring, every quadratic space over $R[T]$ is extended from R. We show in this note that if $R = \mathbf{R}[[X, Y]]$, there exist anisotropic quadratic spaces over $R[T]$ which are not extended from R. This is in contrast to a result of Mohan Kumar and Lindel in the linear case [2, Theorem 5.1, p. 150].

Let $R = \mathbf{R}[[X, Y]]$, R denoting the field of real numbers. Let $\mathbf{H}$ be the quaternions over $\mathbf{R}$ and $A = \mathbf{H}[[X, Y]]$ the ring of formal power series in the variables X and Y over $\mathbf{H}$. We have an $A[T]$-linear map $A[T]^2 \xrightarrow{\eta} A[T]$ defined by $(1, 0) \to XT + i$, $(0, 1) \to YT + j$ which is clearly a surjection. Let P be the kernel of η.

PROPOSITION. *The module P is nonfree projective over $A[T]$.*

PROOF. The first projection of $A[T]^2$ onto $A[T]$ maps P isomorphically onto the left ideal $\mathfrak{A}$ of $A[T]$ generated by $1 + Y^2T^2$ and $1 + (iX + jY)T - kXYT^2$ [4, p. 143]. We prove that P is not free by showing that $\mathfrak{A}$ is not principal. We note first that $\mathfrak{A}$ is not the unit ideal since it is generated modulo X by $1 + jYT$ which is not a unit in $\mathbf{H}[[Y]][T]$. Suppose $\mathfrak{A}$ is principal, generated by f. Then $\deg_T f < 2$. If $\deg_T f = 0$, then it follows that $\mathfrak{A}$ is the unit ideal, which is not the case. If $\deg_T f = 2$, then $1 + Y^2T^2$ and $1 + (iX + jY)T - kXYT^2$ are unit left multiples of each other, which is again not possible. Let $\deg_T f = 1$ and $f = a + bT$, $a, b \in \mathbf{H}[[X, Y]]$. Then a is a unit and we assume $a = 1$ so that $f = 1 + bT$, $b \in \mathbf{H}[[X, Y]]$. We then have

$$1 + Y^2T^2 = (1 + cT)(1 + bT),$$

$$1 + (iX + jY)T - kXYT^2 = (1 + dT)(1 + bT).$$

From the first equation we get $c = -b$, $-b^2 = Y^2$. This implies that $b = \lambda Y$, $\lambda \in \mathbf{H}[[X, Y]]$. From the second equation we get $d + b = iX + jY$, $db = -kXY$ so that we have

$$(iX + jY)\lambda = -(kX + Y).$$

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If $\lambda = \lambda_0 + \lambda_1 X + \lambda_2 Y + \ldots, \lambda_i \in \mathbf{H}$, we have $i\lambda_0 = -k$, $j\lambda_0 = -1$, a contradiction, which proves the proposition.

The reduced norm on P [**1**, Theorem 2.1] gives rise to a quadratic space of rank 4 and discriminant 1 over $R[T]$. This space is anisotropic and not extended from R (in fact indecomposable) in view of [**1**, Theorem 4.6].

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