PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, November 1981

ISOMORPHIC GROUP RINGS WITH NONISOMORPHIC COMMUTATIVE COEFFICIENTS

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ABSTRACT. Let X be an infinite cyclic group. An example of two noncommutative nonisomorphic rings R, S such that their group rings RX, SX are isomorphic has been given in [1]. In the present note, we show that there also exist commutative nonisomorphic noetherian domains A, B of Krull dimension 2 such that the group rings AX, BX are isomorphic. That solves Problem 27 of [4] in the negative.

In what follows, some elementary properties of Dedekind domains contained in quadratic extensions of the field of rationals Q will be used (cf. [2]). Let

$$a = \frac{1 + i\sqrt{47}}{2}, \quad D = Z[a], \quad K = Q(a).$$

If φ is the nontrivial automorphism of K then φ is the only nontrivial automorphism of Dedekind domain D and it induces an automorphism of the group of fractional ideals of K. Let I be the ideal in D generated by 2 and a. Then, it may be easily checked that I is not a principal ideal, $I \cap \varphi(I) = 2D$ and $I^5 = bD$, where $b = 4 + a$. Thus, if $\equiv$ means coincidence of classes of fractional ideals, then $\varphi(I) \equiv I^4 \equiv I^{-1}$ and $\varphi(I^{-1}) \equiv I$. Hence, I^2 is not equivalent to $\varphi^m(I^l)$ where $m = 1, 2$; $l = \pm 1$. Now, let

$$A = \bigoplus_{n=-\infty}^{\infty} I^{2n}t^n, \qquad B = \bigoplus_{n=-\infty}^{\infty} I^n t^n \subset K\langle t \rangle$$

where $\langle t \rangle$ is an infinite cyclic group which is generated by t. We shall show that the rings A, B are not isomorphic. Let us suppose σ: $A \to B$ is an isomorphism. Under localization at $S = Z \setminus \{0\}$, σ becomes an isomorphism of $K\langle t \rangle$.

Hence $\sigma(K) = K$ and so $\sigma_{|K} = \varphi^m$, where $m = 1$ or $m = 2$ (cf. [4]). Moreover, $\sigma(t) = ct^l$, $c \in K^*$, $l = \pm 1$. Since $I^l t^l = \sigma(I^2 t) = \sigma(I^2)ct^l$ we have $\sigma(I^2) \equiv I^l$. Thus $\varphi^m(I^2) \equiv I^l$, which is impossible.

Now, we shall establish an isomorphism of the group rings AX, BX. Let τ be the K-automorphism of $K\langle t \rangle$ given by formulas $\tau(t) = t^2x$, $\tau(x) = bt^5x^2$ where x generates the group X. Then τ^{-1} is given by $\tau^{-1}(t) = b^{-1}t^{-2}x$, $\tau^{-1}(x) = b^2t^5x^{-2}$. Since b generates I^5 then it may be computed that $\tau(AX) \subset BX$ and $\tau^{-1}(BX) \subset AX$. Thus, τ induces an isomorphism of the rings AX, BX. Similar examples may be constructed starting from some other Dedekind domains.

It is easily seen that the above-considered rings A, B are finitely generated and hence noetherian. Moreover, the Krull dimension of A, B equal to 2. It is known

Received by the editors August 12, 1980.
1980 *Mathematics Subject Classification*. Primary 16A27.

0002-9939/81/0000-0504/$01.50

that if R, S are rings (not necessarily commutative) with classical Krull dimension 0 and $RX \simeq SX$ then $R \simeq S$. Moreover, there exist noncommutative and nonisomorphic noetherian domains R, S of Krull dimension 1 such that $RX \simeq SX$, [3].

ADDED IN THE PROOF. Recently, a paper *On the coefficient ring of a torus extension*, Osaka J. Math. **17** (1980), 769–782, by K. Yoshida has appeared. Two-dimensional, nonisomorphic affine algebras, torus extensions of which are isomorphic, are constructed.

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