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$L^1(I, X)$ WITH ORDER CONVOLUTION

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ABSTRACT. It is shown that the maximal ideal space of $L^1(I, X)$ is $(0, 1] \times \mathfrak{M}(X)$, where $\mathfrak{M}(X)$ denotes the maximal ideal space of the Banach algebra X. The Gelfand topology on the Carrier space $(0, 1] \times \mathfrak{M}(X)$ coincides with the topology which is the product of the interval topology in $(0, 1]$ and the Gelfand topology on $\mathfrak{M}(X)$. Moreover, the Gelfand transform has the form of an indefinite integral.

Let I denote the interval $[0, 1]$ of real numbers. I is a totally ordered set with the semigroup structure obtained by defining $xy = \max\{x, y\}$. When I is provided with the usual interval topology, I is a compact topological semigroup. Let $C(I)$ denote the linear space of all complex-valued continuous functions on I. We give $C(I)$ the usual norm

$$\|f\| = \max_{x \in I} |f(x)|$$

for f in $C(I)$. Let $C(I)^*$ denote the conjugate space of $C(I)$, that is, the linear space of all continuous complex-valued linear functionals L on $C(I)$. It is well known that each $L \in C(I)^*$ has a unique representation as an integral with respect to a complex-valued, countably additive, regular measure λ defined on all Borel subsets of I [**10**, p. 364]. That is,

$$L(f) = \int_I f(x)\, d\lambda(x)$$

for all f in $C(I)$.

Let X be a commutative Banach algebra with identity e, $\|e\| = 1$. Denote by $M(I, X)$ the set of all countably additive, regular vector-valued measures defined on the σ-algebra $\mathfrak{B}(I)$ of Borel sets in I with values in X, which have finite total variation [**3**]. With the total variation as norm, $\|m\| = |m|(I)$, $m \in M(I, X)$, $M(I, X)$ is a Banach space ([**6**, p. 161] or [**8**, p. 103]).

Following [**3**, p. 379] a linear operation U: $C(I) \to X$ is said to be dominated if there exists a regular positive Borel measure ν such that

$$\|U(f)\| < \int_I |f|\, d\nu$$

for every $f \in C(I)$; one says that U is dominated by ν or that ν dominates U. Then there exists a least positive regular Borel measure dominating U, denoted by μ_U.

Put

$$\|U\| = \sup\Big\|\sum U(f_i)\Big\| \quad \text{and} \quad |||U||| = \sup \sum \|U(f_i)\|$$

where the supremum is taken over all finite families $\{f_i\}$ of $C(I)$ with $\sum \|f_i\| < 1$.

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We shall make use of the following result proved in [3, p. 380].

THEOREM 1. *There exists an isomorphism $U \leftrightarrow m$ between the set of the dominated linear operations U: $C(I) \to X$ and the set of the regular Borel measures m: $\mathfrak{B}(I) \to X$ with finite variation μ, given by the equality*

$$U(f) = \int_I f\,dm \quad \text{for every } f \in C(I).$$

Moreover, if U and m are in correspondence, we have

$$\mu_U = \mu, \quad \|U\| = \|m\|(I) \quad \text{and} \quad |||U||| = |m|(I) = \mu(I).$$

Recall that $\|m\|(I)$ is the semivariation of m on I, that is,

$$\|m\|(I) = \sup\left\|\sum a_i m(E_i)\right\|$$

where the supremum is taken over all finite families $\{E_i\}$ of disjoint sets of $\mathfrak{B}(I)$ with $\bigcup_i E_i = I$ and for all the finite families $\{a_i\}$ of complex numbers such that $|a_i| \leqslant 1$ for each i.

Duchon [4] has introduced the convolution algebra $M(I, X)$ as follows: Let f be in $C(I)$. Then the function of two variables $f(xy)$ is continuous in $I \times I$. Let m, n be in $M(I, X)$. Let $m \otimes n$ be a regular Borel vector-valued measure with finite variation on $\mathfrak{B}(I \times I)$ that is an extended product of m and n (see [5, p. 1469]). Then $m * n$ is the measure determined by the dominated operation W: $C(I) \to X$ given by the equality

$$\begin{aligned} W(f) &= \int_{I\times I} f(xy)\, d(m\otimes n)(x,y) \\ &= \int_I\left\{\int_I f(xy)\, dm(y)\right\} dn(x) = \int_I f\, d(m*n) \end{aligned} \tag{1}$$

for every f in $C(I)$. Moreover,

$$\|m * n\| \leqslant \|m\|\,\|n\|, \qquad m, n \in M(I, X).$$

With this product, $M(I, X)$ is a Banach algebra, which is commutative if X is. The structure of measure algebras $M(G, X)$, where G is a totally ordered compact topological semigroup with multiplication in G defined by $xy = \max\{x, y\}$, is discussed in [5]. If $\mathfrak{M}(X)$ denotes the space (in the usual weak topology) of regular maximal ideals in X and F_M represents the canonical homomorphism from X onto the complex numbers $\mathbf{C}$ associated with an $M \in \mathfrak{M}(X)$, then for every homomorphism π of $M(I, X)$ onto $\mathbf{C}$, there exist a $b \in I$ and $M \in \mathfrak{M}(X)$ such that $\pi(m) = F_M m([0, b])$ for all $m \in M(I, X)$ or there exist a $b \in (0, 1]$ and $M \in \mathfrak{M}(X)$ such that $\pi(m) = F_M m([0, b))$ for all $m \in M(I, X)$. Conversely, if $a \in [0, 1]$ ($a \in (0, 1]$), $M \in \mathfrak{M}(X)$, then the mapping $m \to F_M m([0, a])$ ($m \to F_M m([0, a))$) is a homomorphism of $M(I, X)$ onto $\mathbf{C}$.

In what follows we show that $L^1(I, X)$ is a subalgebra of $M(I, X)$ and study the structure of this algebra. The maximal ideal space of $L^1(I, X)$ corresponds to $(0, 1] \times \mathfrak{M}(X)$. The Gelfand topology coincides with the topology which is the product of the interval topology on (0, 1] and the weak topology on $\mathfrak{M}(X)$ and the Gelfand transform has the form of the indefinite integral.

For terms pertaining to vector measures and Bochner integral, the general reference is [2]. We list the following definition:

DEFINITION 2. Let (Ω, Σ, μ) be a finite measure space. A Banach space X has the Radon-Nikodym property with respect to (Ω, Σ, μ) if for each μ-continuous vector measure m: $\Sigma \to X$ of bounded variation there exists g in $L^1(\mu, X)$ such that $m(E) = \int_E g\, d\mu$ for all $E \in \Sigma$.

Throughout the discussion we shall assume that the Banach algebra X has the Radon-Nikodym property with respect to the Lebesgue measure on [0, 1].

1. $L^1(I, X)$ as a subalgebra of $M(I, X)$. Every function $f \in L^1(I, X)$ determines a measure $\mu_f \in M(I, X)$ where

$$\mu_f(E) = \int_E f(x)\, dx.$$

The measure μ_f is absolutely continuous with respect to Lebesgue measure and, in view of the Radon-Nikodym property of X with respect to Lebesgue measure, every measure which is absolutely continuous with respect to Lebesgue measure is of this form. Since functions in $L^1(I, X)$ which are equal almost everywhere are identified, different functions determine different measures. Moreover, we have $\|\mu_f\| = \|f\|_1$. We can thus view $L^1(I, X)$ as a linear subspace of $M(I, X)$. Our first result is that this linear subspace is in fact a subalgebra.

We shall need a notation for certain subsets of I. We put $A_1 = \{x, y \in I: x < y\}$ and $A_2 = \{x, y \in I: x > y\}$.

THEOREM 3. *If $f, g \in L^1(I, X)$, then $f * g \in L^1(I, X)$ and*

$$(f * g)(x) = f(x)\int_0^x g(y)\, dy + g(x)\int_0^x f(y)\, dy \quad a.e. \tag{2}$$

PROOF. We will show that the function defined on the right side of (2) belongs to $L^1(I, X)$. We have

$$\begin{aligned}
&\int_0^1 \left\| f(x)\int_0^x g(y)\, dy + g(x)\int_0^x f(y)\, dy \right\| dx \\
&\quad < \int_0^1\int_0^1 \|f(x)\|\,\|g(y)\|\chi_{A_2}\, dy\, dx + \int_0^1\int_0^1 \|g(x)\|\,\|f(y)\|\chi_{A_2}\, dy\, dx \\
&\quad = \int_0^1\int_0^1 \|f(x)\|\,\|g(y)\|\chi_{A_2}\, dy\, dx + \int_0^1\int_0^1 \|f(x)\|\,\|g(y)\|\chi_{A_1}\, dy\, dx \\
&\quad = \int_0^1\int_0^1 \|f(x)\|\,\|g(y)\|(\chi_{A_1} + \chi_{A_2})\, dy\, dx \\
&\quad = \int_0^1\int_0^1 \|f(x)\|\,\|g(y)\|\, dy\, dx = \|f\|_1\,\|g\|_1.
\end{aligned}$$

Thus

$$f(x)\int_0^x g(y)\, dy + g(x)\int_0^x f(y)\, dy \in L^1(I, X).$$

To determine the convolution product of f and g, let h be an arbitrary function in $C(I)$. Then according to (1),

$$\begin{aligned}\int_I h(z)\,d(\mu_f * \mu_g)(z) &= \int_0^1\int_0^1 h(xy)f(x)g(y)\,dx\,dy\\ &= \int_0^1\int_0^1 [\chi_{A_1}h(y) + \chi_{A_2}h(x)]f(x)g(y)\,dx\,dy\\ &= \int_0^1 h(y)g(y)\left[\int_0^1 \chi_{A_1}f(x)\,dx\right]dy + \int_0^1 h(x)f(x)\left[\int_0^1 \chi_{A_2}g(y)\,dy\right]dx\\ &\qquad\qquad \text{(using Fubini's theorem [3, Theorem 4, p. 1469])}\\ &= \int_0^1 h(y)\left[g(y)\int_0^y f(x)\,dx\right]dy + \int_0^1 h(y)\left[f(y)\int_0^y g(x)\,dx\right]dy\\ &= \int^1 h(y)\left[g(y)\int_0^y f(x)\,dx + f(y)\int_0^y g(x)\,dx\right]dy.\end{aligned}$$

Thus, in view of Theorem 1, the measure $\mu_f * \mu_g$ agrees with the measure determined by the function $f(x)\int_0^x g(y)\,dy + g(x)\int_0^x f(y)\,dy$.

From the above theorem we have that $L^1(I, X)$ is a subalgebra of $M(I, X)$ and, from the results of [4], $M(I, X)$ is semisimple if and only if X is semisimple. Thus we have the following result:

COROLLARY 4. *$L^1(I, X)$ with order convolution is a commutative semisimple Banach algebra iff X is semisimple.*

REMARK. $L^1(I, X)$ is not an ideal in $M(I, X)$. In fact, the function $e\chi_{[c,d]}$, where c and d are distinct interior points of I, when convolved with the mass e at d gives the mass $(d - c)e$ at the point d. Of course the set of discrete measures in $M(I, X)$, $l_1(I, X)$, is a subalgebra of $M(I, X)$. It is easy to check that $l_1(I, X) \oplus L^1(I, X)$, that is those measures in $M(I, X)$ of the form $\mu_d + \mu_f$ where μ_d is discrete with values in X and $f \in L^1(I, X)$, is also a subalgebra of $M(I, X)$.

2. The Gelfand representation. The Gelfand theory for commutative Banach algebras provides a representation of these algebras as algebras of continuous functions on a locally compact space. We show that this representation for $L^1(I, X)$ with order convolution is given by the indefinite integral.

Since $L^1(I, X)$ is a subalgebra of $M(I, X)$ each homomorphism of $M(I, X)$ into the complex numbers will be a homomorphism of $L^1(I, X)$ into the complex numbers. Thus for each $b \in I$ ($b \in (0, 1]$) and $M \in \mathfrak{M}(X)$ the mappings

$$f \to F_M \int \chi_{[0,b]}(x)f(x)\,dx \quad \text{and} \quad f \to F_M \int \chi_{[0,b)}(x)f(x)\,dx$$

are homomorphisms of $L^1(I, X)$ into the complex numbers. However, since $\chi_{[0,b]} = \chi_{[0,b)}$ almost everywhere, they define the same homomorphism of $L^1(I, X)$ into $\mathbf{C}$. There is the possibility too that there are homomorphisms on $L^1(I, X)$ which are not of this form. The following theorem combined with the above remarks gives a complete characterisation of nonzero homomorphisms of $L^1(I, X)$ into complex numbers.

THEOREM 5. *Every homomorphism ϕ of $L^1(I, X)$ onto the complex numbers is of the form*

$$\phi(f) = F_M \int_0^\alpha f(x)\,dx \tag{3}$$

for some $0 < \alpha \leqslant 1$ and F_M, where F_M represents the canonical homomorphism from X onto $\mathbf{C}$ associated with an $M \in \mathfrak{M}(X)$.

PROOF. Since simple functions are dense in $L^1(I, X)$ and ϕ is not a zero homomorphism of $L^1(I, X)$, there exists $\chi_{E_0}x$ such that $\phi(\chi_{E_0}x) \neq 0$. Define

$$h(fe) = \phi(fe * \chi_{E_0}x)/\phi(\chi_{E_0}x), \qquad f \in L^1(I).$$

If $\chi_E y$ is such that $\phi(\chi_E y) \neq 0$, then

$$\phi(fe * \chi_{E_0}x)/\phi(\chi_{E_0}x) = \phi(fe * \chi_E y)/\phi(\chi_E y).$$

One need only cross multiply and observe that ϕ is multiplicative and the algebra is commutative. Moreover, h is a homomorphism. In fact,

$$\begin{aligned} h(fe * ge) &= \frac{\phi(fe * ge * \chi_{E_0}x)}{\phi(\chi_{E_0}x)} = \frac{\phi(fe * ge * \chi_{E_0}x)}{\phi(\chi_{E_0}x)} \cdot \frac{\phi(\chi_{E_0}x)}{\phi(\chi_{E_0}x)} \\ &= \frac{\phi(fe * ge * \chi_{E_0}x * \chi_{E_0}x)}{\phi(\chi_{E_0}x)\phi(\chi_{E_0}x)} = \frac{\phi(fe * \chi_{E_0}x)}{\phi(\chi_{E_0}x)} \cdot \frac{\phi(ge * \chi_{E_0}x)}{\phi(\chi_{E_0}x)} = h(fe)h(ge). \end{aligned}$$

Since $L^1(I)e$ is isometrically isomorphic to $L^1(I)$, it follows, in view of [12], that there exists α, $0 < \alpha \leqslant 1$, such that $h(fe) = \int_0^\alpha f(x)\,dx$. Define

$$k(y) = \phi(\chi_{E_0}xy)/\phi(\chi_{E_0}x), \qquad y \in X.$$

It may be easily checked that k is well defined and is a multiplicative linear homomorphism of X. So there exists $M \in \mathfrak{M}(X)$ such that $k(y) = F_M(y)$ for all $y \in X$.

$$\begin{aligned} h(\chi_E e)k(y) &= \frac{\phi(\chi_{E_0}e * \chi_{E_0}x)}{\phi(\chi_{E_0}x)} \cdot \frac{\phi(\chi_{E_0}xy)}{\phi(\chi_{E_0}x)} \\ &= \frac{\phi(\chi_E y * \chi_{E_0}x * \chi_{E_0}x)}{\phi(\chi_{E_0}x * \chi_{E_0}x)} = \phi(\chi_E y). \end{aligned}$$

Thus

$$\phi(\chi_E y) = F_M(y)\int_0^\alpha \chi_E(t)\,dt = F_M\left(\int_0^\alpha y\chi_E(t)\,dt\right).$$

Suppose now f is an arbitrary function in $L^1(I, X)$. There exists a sequence of simple functions $\{f_n\}$ such that $\mathrm{Lim}_n\|f_n - f\| = 0$ and $\mathrm{Lim}_n \int_0^1 \|f_n - f\|\,dx = 0$. Moreover, $\phi(f_n) = F_M \int_0^\alpha f_n(t)\,dt$. By the dominated convergence theorem and the continuity of F_M, we have

$$\phi(f) = F_M \int_0^\alpha f(t)\,dt.$$

We can thus identify the Carrier space of $L^1(I, X)$ with $(0, 1] \times \mathfrak{M}(X)$, and the Gelfand transform of f in $L^1(I, X)$ is the indefinite integral. The following theorem shows that the Gelfand topology in the Carrier space $(0, 1] \times \mathfrak{M}(X)$ of $L^1(I, X)$ is the product of the interval topology on $(0, 1]$ and the weak topology on $\mathfrak{M}(X)$.

THEOREM 6. *The Gelfand topology, τ, on $(0, 1] \times \mathfrak{M}(X)$ coincides with the topology τ', which is the product of the interval topology on $(0, 1]$ and the weak topology on $\mathfrak{M}(X)$.*

PROOF. The Gelfand topology, τ, is the weakest topology for which the functions

$$\hat{f}(\alpha, M) = \int_0^\alpha F_M(f(x))\, dx$$

are continuous. Since these functions are continuous with respect to τ', τ is weaker than τ'. The functions $\hat{f}(\alpha, M)$ clearly separate the points of $(0, 1] \times \mathfrak{M}(X)$, vanish at infinity, and do not all vanish at a particular point in $(0, 1] \times \mathfrak{M}(X)$. Thus the weak topology τ induced on $(0, 1] \times \mathfrak{M}(X)$ by these functions coincides with τ' [**13**, p. 12].

REMARK. The result about the Carrier space of $L^1(I, X)$ can also be obtained by using a theorem of Grothendieck [**9**, Chapter 1, p. 58] and a theorem of Gelbaum [**7**]. However, our method is very simple and straightforward.

It follows from the above results that $L^1(I, X)$ has no identity. It is clear that the adjunction of an identity to L^1 is equivalent to the adjunction of the mass e at the point 0 to the algebra $L^1(I, X)$. However, there are approximate identities in $L^1(I, X)$.

THEOREM 7. *Given $f \in L^1(I, X)$ and $\varepsilon > 0$ there exists $t \in I$ such that if $\nu(x) = u(x)e$, where $u(x)$ is any nonnegative function in $L^1(I)$ which vanishes to the right of t, and $\int_0^1 \nu(x)\, dx = e$, then $\|f - \nu * f\|_1 \leqslant \varepsilon$.*

PROOF. Choose $t > 0$ such that $\int_0^t \|f(x)\|\, dx < \varepsilon/3$. If ν satisfies the conditions of the theorem, then for $x > t$, $(\nu * f)(x) = f(x)$. Thus

$$\begin{aligned}
\|\nu * f - f\| &= \int_0^t \left\| \nu(x) \int_0^x f(y)\, dy + f(x) \int_0^x \nu(y)\, dy - f(x) \right\| dx \\
&\leqslant \int_0^t \left[u(x) \int_0^x \|f(y)\|\, dy + \|f(x)\| \int_0^x u(y)\, dy + \|f(x)\| \right] dx \\
&\leqslant \int_0^t u(x) \left[\int_0^x \|f(y)\|\, dy \right] dx + \int_0^t \|f(x)\|\, dx + \int_0^t \|f(x)\|\, dx \\
&\leqslant \int_0^t \frac{\varepsilon}{3} u(x)\, dx + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leqslant \varepsilon.
\end{aligned}$$

REMARKS. (i) If $X = \mathbf{C}$, the field of complex numbers, then $L^1(I, X)$ is the algebra studied in [**12**].

(ii) If $X = L^1(I)$, then it can be shown, using an argument similar to the one given in [**11**], that $L^1(I, L^1(I))$ is isometrically isomorphic to the algebra $L^1(I \times I)$ studied in [**1**].

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