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A NOTE ON INTERTWINING M-HYPONORMAL OPERATORS

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ABSTRACT. If $AX = XB^*$ with A and B M-hyponormal, then $A^*X = XB$. Furthermore, $(\operatorname{ran} X)^-$ reduces A, $\ker X$ reduces B, and $A|(\operatorname{ran} X)^-$ and $B^*|\ker^{\perp} X$ are unitarily equivalent normal operators. An asymptotic version is also proved.

Let $\mathcal{H}$ be a Hilbert space. A bounded operator A on $\mathcal{H}$ is called dominant by J. Stampfli and B. Wadhwa [4] if, for all complex λ, $\operatorname{range}(A - \lambda) \subseteq \operatorname{range}(A - \lambda)^*$, or, equivalently, if there is a real number $M_\lambda \geqslant 1$ such that

$$\|(A - \lambda)^* f\| \leqslant M_\lambda \|(A - \lambda) f\|$$

for all f in $\mathcal{H}$. If there is a constant M such that $M_\lambda \leqslant M$ for all λ, A is called M-hyponormal, and if $M = 1$, A is hyponormal.

Stampfli and Wadhwa showed in [4, Theorem 1] that if A is dominant, B is hyponormal, X is one-to-one and has dense range, and if $AX = XB^*$, then A and B are normal. M. Radjabalipour improved this result by allowing B to be M-hyponormal [3, Theorem 3(a)]. Of course, the condition that A and B are normal allows one to conclude immediately by the usual Putnam-Fuglede theorem that $A^*X = XB$. S. K. Berberian [2] has obtained the latter result under the conditions that A and B are hyponormal and X is Hilbert-Schmidt (but not one-to-one or with dense range). It seems to have escaped notice, however, that if A and B are both M-hyponormal, the conclusion that $A^*X = XB$ can be reached with no restrictions on X at all; moreover, by employing both intertwining equations one can determine precisely the subspaces on which A and B must be normal. We will need two other results from [3]:

THEOREM A (RADJABALIPOUR). *Let A be dominant and let $\mathfrak{M}$ be an invariant subspace of A for which $A|\mathfrak{M}$ is normal. Then $\mathfrak{M}$ reduces A.*

THEOREM B (RADJABALIPOUR). *If A and A^* are M-hyponormal then A is normal.*

We begin with a symmetric version.

THEOREM 1. *Let A be M-hyponormal and suppose that $AX = XA^*$. Then $A^*X = XA$.*

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PROOF. Let $X = H + iJ$ be the Cartesian decomposition of X. By taking the adjoint of the intertwining equation, we obtain $AX^* = X^*A^*$ and thus $AH = HA^*$ and $AJ = JA^*$.

Let $\mathfrak{M}$ be the kernel of H and decompose the Hilbert space as $\mathfrak{M}^\perp \oplus \mathfrak{M}$. $\mathfrak{M}$ is clearly invariant for A^* and we can represent A and H as operator matrices:

$$A = \begin{pmatrix} C & D \\ 0 & E \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}.$$

C is M-hyponormal and since $AH = HA^*$ we have $CK = KC^*$ and because K is one-to-one and has dense range we conclude that C is normal by [3, Theorem 3(a)]. By Theorem A, $D = 0$ and it follows that $A^*H = HA$. Similarly, $A^*J = JA$ and thus $A^*X = XA$.

THEOREM 2. *If A and B are M-hyponormal and $AX = XB^*$ then $A^*X = XB$.*

PROOF. Let

$$\hat{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad \hat{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

$\hat{A}$ is M-hyponormal and $\hat{A}\hat{X} = \hat{X}\hat{A}^*$ and Theorem 1 yields the desired result.

The next theorem, which generalizes Theorem 3(a) of [3], identifies the subspaces on which A and B must be normal.

THEOREM 3. *Let A, B and X be as in Theorem 2. Then*

(a) $(\operatorname{ran} X)^-$ *reduces A and* $\ker X$ *reduces B.*

(b) $A|(\operatorname{ran} X)^-$ *and* $B^*|\ker^\perp X$ *are unitarily equivalent normal operators.*

PROOF. (a) By Theorem 2, $AXX^* = XB^*X^* = XX^*A$. Thus A commutes with XX^* and so $(\operatorname{ran} X)^- = (\operatorname{ran} XX^*)^-$ reduces A. Similarly B commutes with X^*X and $\ker X = \ker X^*X$ reduces B.

(b) Let $X = UP$ be the polar decomposition of X. Since B commutes with P as above, we have

$$(AU - UB^*)P = 0.$$

Let $\mathcal{K}_1 = \ker^\perp X = \ker^\perp P$ and let $\mathcal{K}_2 = (\operatorname{ran} X)^-$; let $A_2 = A|\mathcal{K}_2$ and $B_1 = B|\mathcal{K}_1$. Let $V\colon \mathcal{K}_1 \to \mathcal{K}_2$ be defined by $Vf = Uf$ for all $f \in \mathcal{K}_1$. The equation above then becomes

$$A_2V = VB_1^*.$$

Since V is an invertible isometry we have that A_2 is unitarily equivalent to B_1^* and since A_2 and B_1 are both M-hyponormal, Theorem B implies that both are normal. The proof is complete.

We now proceed to an asymptotic version, which is most readily attained by employing some machinery developed by Berberian [1]. We sketch Berberian's construction here; the details are in [1]. Let $\mathcal{K}$ be a Hilbert space and let $\mathfrak{B}$ be the set of bounded sequences of vectors $\{f_n\}$, with $f_n \in \mathcal{K}$. Let "glim" denote a generalized limit defined on the collection of bounded sequences of complex numbers, and let $\mathfrak{N} = \{\{f_n\} \in \mathfrak{B}\colon \operatorname{glim}\{\|f_n\|\} = 0\}$. Then the set $\mathfrak{P} = \mathfrak{B}/\mathfrak{N}$ has a pre-Hilbert space structure with the inner product $(\{f_n\} + \mathfrak{N}, \{g_n\} + \mathfrak{N}) = \operatorname{glim}(f_n, g_n)$. The map $f \to \{f, f, \ldots\} + \mathfrak{N}$ is a natural imbedding of $\mathcal{K}$ into $\mathfrak{P}$.

Let $\mathcal{H}$ be the completion of $\mathcal{P}$. If $\{T_n\}$ is a bounded sequence of operators on $\mathcal{H}$ and if $\{f_n\} \in \mathcal{N}$, then the sequence $\{T_n f_n\} \in \mathcal{N}$ and it follows that the function that maps $\{g_n\} + \mathcal{N}$ to $\{T_n g_n\} + \mathcal{N}$ defines a bounded linear operator on $\mathcal{H}$ which we call $\phi(\{T_n\})$. It is easy to check that $\phi(\{T_n\}) = 0$ if and only if $\|T_n\| \to 0$, that $\phi(\{T_n^*\}) = \phi(\{T_n\})^*$, and that $\phi(\{T_n\})$ is positive if and only if $T_n - |T_n| \to 0$, in the strong operator topology.

THEOREM 4. *Let $\{T_n\}$ and $\{S_n\}$ be bounded sequences for which there exists a number M such that, for all complex numbers λ,*

$$M^2(T_n - \lambda)^*(T_n - \lambda) - (T_n - \lambda)(T_n - \lambda)^* - |M^2(T_n - \lambda)^*(T_n - \lambda) - (T_n - \lambda)(T_n - \lambda)^*| \to 0 \quad (\textit{strongly})$$

and

$$M^2(S_n - \lambda)^*(S_n - \lambda) - (S_n - \lambda)(S_n - \lambda)^* - |M^2(S_n - \lambda)^*(S_n - \lambda) - (S_n - \lambda)(S_n - \lambda)^*| \to 0 \quad (\textit{strongly}).$$

Let $\{X_n\}$ be a bounded sequence and suppose that $T_nX_n - X_nS_n^* \to 0$. Then $T_n^*X_n - X_nS_n \to 0$.

PROOF. The conditions on $\{T_n\}$ and $\{S_n\}$ imply that $\phi(\{T_n\})$ and $\phi(\{S_n\})$ are M-hyponormal. The equation

$$\phi(\{T_n\})\phi(\{X_n\}) = \phi(\{X_n\})\phi(\{S_n\})^*$$

holds, and Theorem 2 yields the result.

COROLLARY. *If T and S are M-hyponormal and if $TX_n - X_nS^* \to 0$ for a bounded sequence $\{X_n\}$, then $T^*X_n - X_nS \to 0$ as well.*

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