SOME INTRINSIC COORDINATES ON TEICHMÜLLER SPACE

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Abstract. We give a new construction of intrinsic global coordinates on the Teichmüller space $T_p$ of closed Riemann surfaces of genus $p > 2$. Our construction produces an injective holomorphic map from $T_p$ into the space of Schottky groups of genus $p$.

1. Introduction. Since the Teichmüller space $T_p$ of closed Riemann surfaces of genus $p > 2$ is a complex analytic manifold of dimension $n = 3p - 3$, any injective holomorphic map $f: T_p \rightarrow \mathbb{C}^n$ defines a set of global coordinate functions on $T_p$. We call these coordinates intrinsic if the coordinates $f(t)$ are determined from the marked Riemann surface $t$ alone and do not depend on the choice of a basepoint $t_0$ in $T_p$. In this paper we describe a new way to define intrinsic coordinates on $T_p$.

We should emphasize that we are defining complex coordinates for the complex manifold $T_p$. The problem of finding real analytic coordinates was solved classically with the help of Fuchsian groups. The first global complex coordinates were found by Bers [2], using quasi-fuchsian groups. The Bers coordinates depend on the choice of a basepoint. Maskit [5] defined the first intrinsic (complex) coordinates. Our coordinates are closer in spirit to the Bers coordinates since we use quasi-fuchsian groups. It would, of course, be interesting to find global coordinates for $T_p$ that do not depend on uniformization by Kleinian groups.

2. Quasifuchsian groups. Let $\Gamma$ be a quasifuchsian group of type $(p, 0)$. This means that the limit set $\Lambda(\Gamma)$ is a Jordan curve in the extended plane, that $\Gamma$ maps each of the Jordan regions $D_1$ and $D_2$ bounded by $\Lambda(\Gamma)$ into itself, and that the quotient maps $D_1 \rightarrow D_1/\Gamma$ and $D_2 \rightarrow D_2/\Gamma$ are unramified coverings of closed Riemann surfaces of genus $p$.

Lifting a canonical dissection of the surface $D_1/\Gamma$ to $D_1$, we can choose an ordered $2p$-tuple

$$\sigma = (A_1, B_1, A_2, B_2, \ldots, A_p, B_p)$$

of Möbius transformations such that the $A_j$ and $B_j$ generate $\Gamma$ and satisfy the relation

$$\prod_{j=1}^{p} C_j = I, \quad C_j = A_j B_j A_j^{-1} B_j^{-1}. \quad (2)$$

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The pair \((\sigma, \Gamma)\) is called a marked quasifuchsian group. We say that \((\sigma, \Gamma)\) is normalized if the attractive fixed points of \(B_1\) and \(B_2\) and the repulsive fixed point of \(B_1\) are at 0, 1, and \(\infty\) respectively. Our reason for that normalization will become clear in §5. It is well known (see [3]) that the space of normalized groups \((\sigma, \Gamma)\) is a complex manifold, biholomorphically equivalent to \(T_p \times T_p\).

To represent \(T_p\) by a set of normalized groups one must embed \(T_p\) in \(T_p \times T_p\). The Bers coordinates are obtained by identifying \(T_p\) with a “slice” \(T_p \times \{t_0\}\) in \(T_p \times T_p\). Our method is to identify \(T_p\) with the diagonal. Our main theorem gives a general procedure for making that identification, and in §5 we illustrate how to use the main theorem to define intrinsic coordinates on \(T_p\).

3. The main theorem. By definition, if \((\sigma, \Gamma)\) is a marked quasifuchsian group the \(2p\)-tuple \(\sigma\) induces a canonical dissection of \(D_1/\Gamma\). The induced dissection of \(D_2/\Gamma\), however, is not canonical, because of orientation. To identify the space of normalized groups with \(T_p \times T_p\) we use a sense-reversing diffeomorphism to make the dissection of \(D_2/\Gamma\) canonical. The following theorem describes the diagonal.

**Theorem 1.** Let \(W\) be a closed Riemann surface of genus \(p > 2\) with a canonical homotopy basis \(a_1, \ldots, b_p\), and let \(\theta\) be an automorphism of \(\pi_1(W)\) induced by a sense-reversing diffeomorphism of \(W\). There is a unique normalized marked quasifuchsian group \((\sigma, \Gamma)\) such that:

(i) the map from \(\pi_1(W)\) to \(\Gamma\) that sends \(a_j\) to \(A_j\) and \(b_j\) to \(B_j\), \(1 \leq j \leq p\), is induced by a conformal map from \(W\) to \(D_1/\Gamma\),

(ii) there is a conformal map \(F: D_2 \rightarrow D_1\) such that

\[
F(\gamma z) = \theta(\gamma)F(z) \quad \text{for all } \gamma \in \Gamma, z \in D_2.
\]

If \(\theta\) has order two, then \(F\) is a Möbius transformation of order two, and \(F\) and \(\Gamma\) generate a Kleinian group whose deformation space is \(T_p\).

Notice that in (ii) we use the isomorphism (i) between \(\pi_1(W)\) and \(\Gamma\) to interpret \(\theta\) as an automorphism of \(\Gamma\). We refer the reader to [3] for a discussion of deformation spaces of Kleinian groups.

4. Proof of Theorem 1. This result is really a corollary of the simultaneous uniformization theorem of Bers [1], but we find it simplest to give a direct proof, modelled on the proof of [1]. First we choose a holomorphic universal covering of \(W\) by the upper half-plane \(U\), identifying \(\pi_1(W)\) with the group \(G\) of deck transformations in the usual way. We normalize \(G\) so that the attractive fixed points of \(b_1\) and \(b_2\) and the repulsive fixed point of \(b_1\) are at 0, 1, and \(\infty\) respectively. By hypothesis there is a sense-reversing diffeomorphism of \(W\) that induces the automorphism \(\theta\). Lifting to \(U\) we get a diffeomorphism \(f: U \rightarrow U\) such that \(f(gz) = \theta(g)f(z)\) for all \(g \in G, z \in U\). Put \(h(z) = f(\bar{z})\) for \(z\) in the lower half-plane \(U^*\). Then \(h\) is a sense-preserving diffeomorphism of \(U^*\) onto \(U\), and

\[
h(gz) = \theta(g)h(z) \quad \text{for all } g \in G, z \in U^*.
\]

Now let \(w: C \rightarrow C\) be a quasiconformal map such that \(w\) fixes the points 0, 1, and \(\infty\), and both \(w\) and \(w \circ h^{-1}\) are conformal in \(U\) (i.e., \(w_z = 0\) in \(U\) and
\[ \frac{w_z}{w_z} = h_z/h_z \text{ in } U^*. \] Put \( \Gamma = wGw^{-1} \), define the isomorphism \( \varphi: G \to \Gamma \) by

\[(4) \quad \varphi(g) = wgw^{-1} \text{ for all } g \in G,\]

and put \( \sigma = (A_1, B_1, \ldots, A_p, B_p) \), \( A_j = \varphi(a_j) \), \( B_j = \varphi(b_j) \), \( 1 < j < p \). Then \( (\sigma, \Gamma) \) is a normalized marked quasifuchsian group with \( D_1 = w(U) \) and \( D_2 = w(U^*) \), and the conformal map \( w: U \to D_1 \) induces a conformal map of \( W \) onto \( D_1/\Gamma \) that satisfies (i). Moreover \( F = whw^{-1}: D_2 \to D_1 \) is conformal, and (3) and (4) give

\[ F\varphi(g)F^{-1} = whgh^{-1}w^{-1} = w\theta(g)w^{-1} = \varphi(\theta(g)) \]

in \( D_1 \), so \( (\sigma, \Gamma) \) satisfies (i) and (ii).

Suppose \( (\sigma', \Gamma') \) is another normalized group that satisfies (i) and (ii) with \( F' : D_2' \to D_1' \) conformal. Write \( \sigma' = (A'_1, \ldots, B'_p) \). Then (i) gives a conformal map \( \gamma: D_1' \to D_1 \), so that in \( D_1 \) we have

\[ HA_jH^{-1} = A'_j, \quad HB_jH^{-1} = B'_j, \quad 1 < j < p. \]

Put \( C = \gamma \) in \( D_1 \) and \( C = (F')^{-1}HF \) in \( D_2 \). Then \( C \) maps the regular set of \( \Gamma \) conformally onto the regular set of \( \Gamma' \) and induces an isomorphism of \( \Gamma \) onto \( \Gamma' \), so the Marden isomorphism theorem [4] implies that \( C \) is a Möbius transformation. The normalization implies that \( C \) is the identity, so \( (\sigma, \Gamma) = (\sigma', \Gamma') \) and \( (\sigma, \Gamma) \) is unique.

Finally, let \( \theta \) have order two. Put \( C = F \) in \( D_2 \) and \( C = F^{-1} \) in \( D_1 \). Then \( C \gamma C^{-1} = \theta(\gamma) \) in both \( D_1 \) and \( D_2 \), so the Marden isomorphism theorem again implies that \( C \) is a Möbius transformation. By construction \( C \) has order two. Since \( F = C \) in \( D_2 \), \( F \) is itself (extendible to) a Möbius transformation of order two. It is clear from (ii) that the group \( H \) generated by \( \Gamma \) and \( F \) is Kleinian, and \( \Gamma \) is the subgroup of index two that maps the region \( D_1 \) onto itself. By §7 of [3], the deformation space of \( H \) is biholomorphically equivalent to \( T_p \). The equivalence is obtained in the natural way. Each point in the deformation space determines a marked quasifuchsian subgroup \( (\sigma, \Gamma) \), which in turn determines a marked Riemann surface \( (D_1/\Gamma, \sigma) \) in \( T_p \). Theorem 1 is proved.

5. Intrinsic global coordinates. To use Theorem 1 we must choose an automorphism \( \theta \). Let \( (\sigma, \Gamma) \) be a marked quasifuchsian group with \( \sigma \) given by (1). Then \( \Gamma \) is the free group on \( A_1, \ldots, B_p \), modulo the relation (2). Put

\[(5) \quad K_0 = I, \quad K_j = \prod_{i=1}^{j} C_i, \quad 1 < j < p,\]

and define \( \theta \) on generators by

\[(6) \quad \theta(A_j) = K_{j-1}A_j^{-1}K_{j-1}^{-1}, \quad \theta(B_j) = K_jB_jK_j^{-1}, \quad 1 < j < p.\]

It is easy to prove by induction on \( j \) that

\[(7) \quad \theta(K_j) = K_j^{-1}, \quad 1 < j < p.\]

The case \( j = p \) of (7) shows that \( \theta \) preserves the relation (2) and does indeed define an automorphism of \( \Gamma \). It is clear from (6) and (7) that \( \theta \) has order two.
Every automorphism of $\Gamma$ is induced by some diffeomorphism of $D_1/\Gamma$, and any diffeomorphism that induces $\theta$ is sense-reversing since on the level of homology $\theta$ fixes each $B_j$ and reverses each $A_j$. We can therefore apply Theorem 1 to $\theta$.

**Theorem 2.** If $\theta$ is defined by (6), the normalized group $(\sigma, \Gamma)$ given by Theorem 1 is determined by $B_1, \ldots, B_p$. The multipliers of the $B_j$, the repulsive fixed points of $B_2, \ldots, B_p$, and the attractive fixed points of $B_3, \ldots, B_p$ are a global coordinate system for $T_p$.

**Proof.** Let $(\sigma, \Gamma)$ be given by Theorem 1. Since $\theta$ has order two, there is a Möbius transformation $F$ such that

$F^2 = I$ and $\theta(\gamma) = F\gamma F^{-1}$ for all $\gamma \in \Gamma$.

Formula (6) implies by induction on $j$ that

$K_j = \theta(B_j \cdots B_1)(B_j \cdots B_1)^{-1}$

$= F(B_j \cdots B_1)F^{-1}(B_j \cdots B_1)^{-1}, \quad 1 \leq j \leq p.$

Taking $j = p$ in (9) we see that $F$ is the unique Möbius transformation of order two that commutes with the loxodromic transformation $B_p \cdots B_1$. Thus the $B_j$ determine $F$ and, by (9), each $K_j$.

Now put

$F_j = FK_{j-1}A_j, \quad 1 \leq j \leq p.$

We claim that $F_j$ is the unique Möbius transformation of order two that commutes with $B_j$. First, $F_j$ is not the identity map since it interchanges the regions $D_1$ and $D_2$. Next,

$F_j^2 = FK_{j-1}A_jFK_{j-1}A_j = \theta(K_{j-1}A_j)K_{j-1}A_j = I,$

by (6), (7), and (8). Finally,

$F_jB_jF_j^{-1} = F_jB_jF_j = \theta(K_{j-1}A_jB_j)K_{j-1}A_j$

$= A_j^{-1}K_{j-1}^{-1}K_jB_jA_j = A_j^{-1}C_jB_jA_j = B_j,$

which proves our claim.

Since the $B_j$ uniquely determine $F$ and the $F_j$, formulas (9) and (10) show that the $B_j$ determine $(\sigma, \Gamma)$. That proves the first statement of Theorem 2. The second statement follows easily. Indeed, the fixed points and multipliers of the $B_j$ are holomorphic functions on the deformation space of the Kleinian group generated by $\Gamma$ and $F$ (see §8 of [3]), hence on $T_p$. Since these fixed points and multipliers determine the $B_j$, and hence $(\sigma, \Gamma)$, we have defined an injective holomorphic map from $T_p$ into $\mathbb{C}^n$. The theorem is proved.

We remark in conclusion that $(B_1, \ldots, B_p)$ generates a Schottky group of genus $p$, and our coordinates on $T_p$ give an injective holomorphic map of $T_p$ into the space of Schottky groups of genus $p$. We will study the geometry of that map in a forthcoming paper.
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