

## ON COMPLETE BIORTHOGONAL SYSTEMS

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**ABSTRACT.** Fundamental to the study of bases in a separable Hilbert space  $H$  is the notion of a *biorthogonal system*. Two sequences  $\{f_n\}$  and  $\{g_n\}$  of elements from  $H$  are said to be biorthogonal if  $(f_n, g_m) = \delta_{nm}$ . A complete sequence that possesses a biorthogonal sequence is called *exact*. Despite the symmetry of the definition of biorthogonality, simple examples show that  $\{f_n\}$  may be exact while  $\{g_n\}$  fails to be exact. For sequences of complex exponentials in  $L^2(-\pi, \pi)$ , the situation is dramatically different—if the sequence  $\{e^{in\theta}\}$  is exact, then its biorthogonal sequence is also exact.

**1. Introduction.** Fundamental to the study of bases in a separable Hilbert space  $H$  is the notion of a biorthogonal system (see, e.g., [2] and [3] and the references therein).

**DEFINITION.** Two sequences  $\{f_n\}$  and  $\{g_n\}$  of elements from  $H$  are said to be *biorthogonal* if  $(f_n, g_m) = \delta_{nm}$ . A sequence that admits a biorthogonal sequence will be called *minimal*.

It is easy to show that a sequence  $\{f_n\}$  is minimal if and only if none of its elements can be approximated by linear combinations of the others. If this is the case, then a biorthogonal sequence will be uniquely determined if and only if  $\{f_n\}$  is *complete*. (This means that the set of all linear combinations  $c_1 f_1 + \cdots + c_n f_n$  is dense in  $H$ , or, equivalently, that the zero vector alone is perpendicular to every  $f_n$ .) A sequence that is both minimal and complete will be called *exact*.

Despite the symmetry of the definition, it is apparent that for a minimal sequence the property of being complete is not inherited by its biorthogonal sequence. (Cf. if  $\{f_n\}$  is a basis, then  $\{g_n\}$  is also a basis [4, p. 29].) Indeed, if  $\{e_n\}$  is an orthonormal basis for  $H$ , then  $\{e_n + e_1\}_{n=2}^\infty$  is complete and has  $\{e_n\}_{n=2}^\infty$  as its biorthogonal sequence. The following generalization is perhaps even more striking.

**EXAMPLE.** For each infinite-dimensional closed subspace  $K$  of  $H$ , there exists a biorthogonal system  $\{f_n\}, \{g_n\}$  such that  $\{f_n\}$  is complete and the closed subspace spanned by  $\{g_n\}$  is  $K$ . Indeed, let  $\{e_n\}$  be an orthonormal basis for  $K$  and  $B$  an orthonormal basis for  $K^\perp$ . We consider the sequence  $\{e_n + y_n\}$  where the  $y_n$  are chosen from  $B$  in such a way that each element of  $B$  appears as a  $y_n$  infinitely many times. It is readily shown that  $\{e_n + y_n\}$  and  $\{e_n\}$  are biorthogonal and that the former sequence is complete.

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The purpose of this note is to show that for minimal sequences of *complex exponentials*  $\{e^{i\lambda_n t}\}$  in  $L^2(-\pi, \pi)$  the situation is dramatically different—the completeness of such a sequence always ensures the completeness of its biorthogonal sequence.

**THEOREM.** *If the sequence of complex exponentials  $\{e^{i\lambda_n t}\}$  is exact in  $L^2(-\pi, \pi)$ , then its biorthogonal sequence is also exact.*

**2. The Paley-Wiener space.** We shall make use of the Paley-Wiener space  $P$  consisting of all entire functions of exponential type at most  $\pi$  that are square-integrable on the real axis. It is clear that  $P$  is a vector space under pointwise addition and scalar multiplication; it is also an inner product space with respect to the inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx.$$

If  $f \in P$ , then the Paley-Wiener theorem shows that

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) e^{izt} dt,$$

with  $\phi \in L^2(-\pi, \pi)$ . Since the Fourier transform is an isometry,  $P$  is a separable Hilbert space, isometrically isomorphic to  $L^2(-\pi, \pi)$ .

Accordingly, problems involving complex exponentials in  $L^2(-\pi, \pi)$  can be examined via their transform image in  $P$ . Thus, for example, the sequence  $\{e^{i\lambda_n t}\}$  is complete in  $L^2(-\pi, \pi)$  if and only if  $\{\lambda_n\}$  is a *set of uniqueness* for  $P$ , i.e., if the relations

$$f \in P \quad \text{and} \quad f(\lambda_n) = 0 \quad (n = 1, 2, 3, \dots)$$

imply that  $f \equiv 0$ . (A detailed account of the Paley-Wiener space and its role in the study of completeness and expansion properties of sets of complex exponentials can be found in [4].)

**3. Proof of the theorem.** It is convenient to describe the biorthogonal sequence in terms of an entire generating function which vanishes at the  $\lambda_n$ . Let  $S$  be the class of entire functions  $f(z)$  of exponential type at most  $\pi$  such that  $f(x)/(1+x^2)^{1/2} \in L^2(-\infty, \infty)$ . The existence of a function  $f \in S$ , equal to zero at every  $\lambda_n$ , but not identically zero, is necessary and sufficient for  $\{e^{i\lambda_n t}\}$  to be minimal [1, p. 419]. Notice that  $f(z)$  can have no zeros different from the  $\lambda_n$ . Otherwise, additional exponentials could be added to the sequence  $\{e^{i\lambda_n t}\}$  without destroying its minimality, and this contradicts its completeness. It follows in the same way that all the zeros of  $f(z)$  must be simple, and so  $f'(\lambda_n)$  is never zero. Furthermore, if  $g(z)$  is any other function in  $S$ , equal to zero at every  $\lambda_n$ , then  $g(z) = Af(z)$ . For if  $\lambda$  is any one of the  $\lambda_n$ , then the function  $f(z)/f'(\lambda)(z-\lambda) - g(z)/g'(\lambda)(z-\lambda)$  belongs to  $P$  (provided  $g \not\equiv 0$ ), equals zero at every  $\lambda_n$ , and so must be identically zero.

By virtue of the Paley-Wiener theorem, we can write

$$(1) \quad \frac{f(z)}{f'(\lambda_n)(z-\lambda_n)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(t) e^{izt} dt,$$

with  $g_n \in L^2(-\pi, \pi)$ . Thus the sequence  $\{\overline{g_n(t)}\}$  is biorthogonal to  $\{e^{i\lambda_n t}\}$ . To prove it is complete, suppose that for some function  $h \in L^2(-\pi, \pi)$ ,

$$\int_{-\pi}^{\pi} h(t)g_n(t) dt = 0 \quad (n = 1, 2, 3, \dots).$$

It is to be shown that  $h(t) = 0$  almost everywhere. Replacing  $h(t)$  by its Fourier series  $h(t) = \sum_{-\infty}^{\infty} a_k e^{ikt}$  ( $\sum_{-\infty}^{\infty} |a_k|^2 < \infty$ ), and then integrating term-by-term, we find

$$(2) \quad \sum_{-\infty}^{\infty} a_k \int_{-\pi}^{\pi} g_n(t) e^{ikt} dt = 0 \quad (n = 1, 2, 3, \dots).$$

Let us suppose first that no  $\lambda_n$  is an integer. Then (1) and (2) show that

$$(3) \quad \sum_{-\infty}^{\infty} a_k \frac{f(k)}{k - \lambda_n} = 0 \quad (n = 1, 2, 3, \dots).$$

Put  $c_k = a_k f(k)/k$  ( $k \neq 0$ ). Since  $f(k)/(k - \lambda_1)$  are the Fourier coefficients of a function in  $L^2(-\pi, \pi)$ ,  $\{f(k)/k\} \in l^2$ , and hence  $\sum |c_k| < \infty$ . This permits us to write  $c = \sum c_k$ . Since

$$a_k \frac{f(k)}{k - \lambda_n} = c_k \frac{k}{k - \lambda_n} = c_k \left( 1 + \frac{\lambda_n}{k - \lambda_n} \right) \quad (k \neq 0),$$

(3) becomes

$$-\frac{a_0 f(0)}{\lambda_n} + c + \lambda_n \sum_{k \neq 0} \frac{c_k}{k - \lambda_n} = 0 \quad (n = 1, 2, 3, \dots).$$

We define a function  $g(z)$  by writing

$$g(z) = -a_0 f(0) \frac{\sin \pi z}{z} + c \sin \pi z + z \sin \pi z \sum_{k \neq 0} \frac{c_k}{k - z} = g_1 + g_2 + g_3.$$

ASSERTION.  $g(z)$  belongs to the class  $S$ . Clearly,  $g_1$  and  $g_2$  belong to  $S$ . Since the set of functions  $\{(\sin \pi(z - k))/\pi(z - k)\}_{-\infty}^{\infty}$  forms an orthonormal basis for  $P$  (take the Fourier transform of  $e^{ikt}$ ) and  $\sum |c_k|^2 < \infty$ , it follows that the series

$$\sum_{k \neq 0} \pi(-1)^{k+1} c_k \frac{\sin \pi(z - k)}{\pi(z - k)} = \sin \pi z \sum_{k \neq 0} \frac{c_k}{k - z}$$

converges in the topology of  $P$ , and hence pointwise, to a function belonging to  $P$ . Thus  $g_3$  belongs to  $S$ . This proves the assertion. Since  $g(\lambda_n) = 0$  ( $n = 1, 2, 3, \dots$ ), we must have  $g(z) = Af(z)$ .

If  $k \neq 0$ , then

$$g(k) = k\pi(-1)^{k+1} c_k = \pi(-1)^{k+1} a_k f(k).$$

If  $k = 0$ , the first term is still equal to the last. Accordingly,

$$(4) \quad A = \pi(-1)^{k+1} a_k \quad \text{for every } k,$$

since  $f(k)$  is never zero (recall that no  $\lambda_n$  is an integer and  $f(z)$  is equal to zero only at the  $\lambda_n$ ). But  $\{a_k\} \in l^2$ , and therefore (4) is possible only if  $a_k = 0$  for every  $k$ .

Thus  $h(t) = 0$  almost everywhere and the biorthogonal sequence  $\{\overline{g_n(t)}\}$  is complete in  $L^2(-\pi, \pi)$ . This proves the theorem under the assumption that no  $\lambda_n$  is an integer.

In the general case, we can choose a suitable real number  $\alpha$  such that  $\lambda_n + \alpha$  is never an integer. Since multiplication by  $e^{i\alpha t}$  is a unitary operator on  $L^2(-\pi, \pi)$ , the sequence  $\{e^{i(\lambda_n + \alpha)t}\}$  is complete and hence its biorthogonal sequence  $\{e^{i\alpha t} \overline{g_n(t)}\}$  is also complete. Thus  $\{\overline{g_n(t)}\}$  is exact.  $\square$

REMARK. The examples in §1 show that the converse of the theorem is false—there are incomplete sequences of complex exponentials that admit complete biorthogonal sequences. Whether *every* such sequence must admit a complete biorthogonal sequence appears to be an open problem.

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