A NOTE ON STRICTLY CYCLIC WEIGHTED SHIFTS

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Abstract. We answer negatively Shields’ Question 15 in [1].

The object of this note is to answer negatively one of the many questions raised by Allen L. Shields in his article [1]. Question 15 of [1] is: If $T$ is a unilateral weighted shift with weights $w_n$ decreasing to 1 and if $\sum_{n=0}^{\infty} \beta(n)^{-2} < \infty$ where $\beta(n) = w_0 w_1 \cdots w_{n-1}$, must $T$ be strictly cyclic? If $\{w_n\}$ decreases to 0, must $T$ be strongly strictly cyclic?

By Proposition 32 of [1, p. 96], to answer the question negatively it will suffice to show that the supremum

\[ \sup_n \sum_{k=0}^{n} \left( \frac{\beta(n)}{\beta(k)\beta(n-k)} \right)^2 \]

is infinite.

For the first part of the question, we let $\{a_j\}$ be any sequence decreasing to 1 and we choose an increasing sequence of positive integers so that

\[ [a_1^n a_2^{n-1} \cdots a_j^{n-j}]^{-2} \left( \frac{a_{j+1}^2}{a_{j+1}^2 - 1} \right) < 2^j \]

and

\[ (1 + n_j) \left( \frac{a_j}{a_1} \right)^{2n_j} > j. \]

Such a choice is clearly possible. We set $w_n = a_1$, if $0 < n < n_1$, and we set $w_n = a_j$, if $n_j - 1 < n < n_j$. In order to verify that our selection works we use (2) to show that $\sum_{n \geq 0} \beta(n)^{-2} < a_1^2/(a_1^2 - 1) + 1$ and (3) to show that the supremum (1) is infinite.

Consider (2) first. We have

\[ \sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} = \sum_{n=0}^{n_1} \frac{1}{\beta(n)^2} + \sum_{j=2}^{\infty} \left( \sum_{n_j - 1 < n < n_j} \frac{1}{\beta(n)^2} \right) \]

where $\beta(n)$ is given by the equations $\beta(n) = a_1^n$ if $n < n_1$, and $\beta(n) = a_1^n a_2^{n-1} \cdots a_j^{n-j}$ if $n_j - 1 < n < n_j$.
Therefore, $\sum_{n=0}^{\infty} \beta(n)^{-2} < \sum_{n=0}^{\infty} a_i^{2n} = a_i^2/(a_i^2 - 1)$, while 

$$
\sum_{n_{j-1} < n < n_j} \frac{1}{\beta(n)^2} = [a_1^{n} a^{n_2-n_1} \ldots a_j^{n_{j-1}-n_{j-2}}]^{-2} \left( \sum_{n_{j-1} < n < n_j} \frac{1}{a_j^{2(n-n_{j-1})}} \right) < \left[ a_1^{n_1} \ldots a_j^{n_{j-1}-n_{j-2}} \right]^{-2} \left( \frac{a_j^2}{a_j^2 - 1} \right) < 2^{j+1}.
$$

This with (4) shows that $\sum_{n=0}^{\infty} \beta(n)^{-2} < a_i^2/(a_i^2 - 1) + 1$.

To show that (1) is violated we show that a lower bound for 

$$\sum_{k=0}^{n_j} \left( \frac{\beta(n_j)}{\beta(k)\beta(n_j-k)} \right)^2$$

is $j$. Consider 

$$\frac{\beta(n_j)}{\beta(k)\beta(n_j-k)} = \frac{w_k \ldots w_{n_j-1}}{w_0 \ldots w_{n_j-(k+1)}};$$

in both top and bottom there are no more than $n_{j-1}$ weights different from $a_j$. Thus, we can say that $\beta(n_j)/\beta(k)\beta(n_j-k) > (a_j/a_{j+1})^{n_j}$, which together with (3) gives (5).

Note that it is actually possible to choose the sequence $\{w_n\}$ strictly decreasing if desired. One might ask if it is possible to choose the weight sequence to be convex.

The second part of the question is answered with a similar construction. Simply choose $\{a_j\}$ decreasing to zero and choose $\{n_j\}$ so that (3) is satisfied. If $\{w_n\}$ is defined as before, then the supremum (1) is infinite, and $T$ is not strictly cyclic.

**References**


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