FUNCTIONS WHICH OPERATE ON THE REAL PART OF
A FUNCTION ALGEBRA

OSAMU HATORI

Abstract. Recently S. J. Sidney [5] has shown that a "highly nonaffine" function h
on an interval cannot operate by composition on the real part of a nontrivial
function algebra. In this paper, we obtain the general result by considering the case
in which h is not "highly nonaffine".

1. Introduction. Let A be a function algebra on a compact Hausdorff space X
and h a nonaffine function on an interval I. We say that h operates by composition
on Re A if h ◦ u ∈ Re A whenever u ∈ Re A has range in I. We consider a
conjecture: If h operates by composition on Re A, then we have A = C(X). The
theorem of J. Wermer [7] is equivalent to the conjecture for h(t) = t^2. A. Bernard
[1] proved the conjecture for h(t) = |t|. S. J. Sidney [5] obtained results for the
cases that h is "highly nonaffine" or continuously differentiable. Our purpose is to
show the following theorem.

Theorem. Suppose that A is a function algebra on a compact Hausdorff space X
and that h is a nonaffine continuous function on an interval I. If h operates by
composition on Re A, then we have A = C(X).

If h is nonaffine on every nondegenerate subinterval of I, then h is "highly
nonaffine". So without loss of generality we may assume that I = [-1, 1], h = 0 on
[-1, 0] and h is not affine on any open subinterval of I containing 0.

Re A is a Banach space with the usual quotient norm

\[ N(u) = \inf\{ \|f\| : f \in A, \text{Re } f = u \}. \]

We denote f|F the restriction of a function f ∈ C(X) to a subset F ⊆ X. For
nonempty disjoint compact subsets F_1 and F_2 of X we denote

\[ (\text{Re } A)_1 = \{ u ∈ C_R(F_1) : \exists \hat{u} ∈ \text{Re } A, \hat{u}|F_1 = u \}, \]

\[ (\text{Re } A)_2 = \{ u ∈ C_R(F_1) : \exists \hat{u} ∈ \text{Re } A, \hat{u}|F_1 = u, \hat{u}|F_2 = 0 \}. \]

For u ∈ (Re A)_1, we define

\[ N_1(u) = \inf\{ N(\hat{u}) : \hat{u} ∈ \text{Re } A, \hat{u}|F_1 = u \}. \]

For u ∈ (Re A)_2, we define

\[ N_2(u) = \inf\{ N(\hat{u}) : \hat{u} ∈ \text{Re } A, \hat{u}|F_1 = u, \hat{u}|F_2 = 0 \}. \]
Then $(\text{Re} \ A)_1$ and $(\text{Re} \ A)_2^2$ are complete with respect to the norms $N_1(\cdot)$ and $N_2^2(\cdot)$, respectively.

2. Lemmas. Let $x$ be a point of $X$. We say that the function $h$ operates weak-boundedly at $x$ if there exist a $\delta > 0$, an $\epsilon > 0$, a compact neighborhood $F_x$ of $x$, and a compact subset $F_0$ of $X$, which is disjoint from $F_x$, with the following property: $h \circ u \in (\text{Re} \ A)_x$ and $N_x(h \circ u) < \epsilon$ for each $u \in (\text{Re} \ A)^0_x$ with $N_x(u) < \delta$.

**Lemma 1.** Suppose that $h$ operates by composition on $\text{Re} \ A$. Then $h$ operates weak-boundedly at each point of $X$ except for at most finitely many points.

**Proof.** Suppose that the lemma fails. Then there exists a countable subset $\{x_n\}$ of $X$ with the following two properties: (1) Each $x_n$ has a compact neighborhood $F_n$ such that $(\text{Cl}(\bigcup_{k\neq n} F_k)) \cap F_n = \emptyset$ for each positive integer $n$. (2) $h$ does not operate weak-boundedly at each point $x_n$. Let $F_{m(n)}$ denote $\text{Cl}(\bigcup_{k\neq n} F_k)$ for each $n$. $(m(n)$ is the index depending on $n).$ Since $h$ does not operate weak-boundedly at $x_n$, there exists a $u_n \in (\text{Re} A)_{m(n)}$ such that $N^{m(n)}(u_n) < 1/2^n$ and $N_n(h \circ u_n) > n$ for each $n$. There exists a $\hat{u} \in \text{Re} A$ such that $\hat{u}_n|F_n = u_n$, $\hat{u}_n|F_{m(n)} = 0$ and $N(\hat{u}_n) < 1/2^n$ for each $n$. So $\sum_{n=1}^{\infty} u_n = \hat{u} \in \text{Re} A$ and $h \circ \hat{u} \in \text{Re} A$ and $\hat{u}|F_n = u_n$ for each $n$. Thus $N(h \circ \hat{u}) > N_n((h \circ \hat{u})|F_n) = N_n(h \circ u_n) > n$ for each $n$. This contradicts $h \circ \hat{u} \in \text{Re} A$.

**Lemma 2.** Let $F_0$ and $F_1$ be nonempty disjoint compact subsets of $X$. Then $(\text{Re} A)_0^1$ is an ultraseparating Banach function space with respect to the norm $N_0^1(\cdot)$.

**Proof.** For each $p \in \beta(N \times F_0)$ the functional $u \mapsto \tilde{u}(p)$ on $C_R(F_0)$ is linear and multiplicative, so there is a unique $x_p \in F_0$ such that $\tilde{u}(p) = u(x_p)$ for all $u \in C_R(X)$. (where $u_n = u$ for all $n$.) Let us take $p, q \in \beta(N \times F_0)$ and $p \neq q$. We shall find $g \in \ell^\infty(N, \text{Re} A)$ such that $g(x) = 0$ for $x \in \beta(N \times F_1)$ and $g(p) \neq g(q)$. We consider the following three cases:

1. $x_p \neq x_q$.
2. $x_p = x_q$, $f(q)$ whenever $f \in \ell^\infty(N, \text{Re} A)$ vanishes on $N \times \{x_p\}$.
3. $x_p = x_q$, there exists an $\tilde{f} \in \ell^\infty(N, \text{Re} A)$ such that $\tilde{f}$ vanishes on $N \times \{x_p\}$ and $(\tilde{f}(p) \neq \tilde{f}(q))$.

Case (1). Since $\text{Re} A$ is uniformly dense in $C_R(X)$, there exists an $f \in \text{Re} A$ such that $-1 < f(x_p) < 0$, $f(F_1) \subset [-1, 0]$ and $f(x_q) \notin h^{-1}(0)$. Then $g = \tilde{h} \circ \tilde{f} = (h \circ \tilde{f}(x_p)$ where $u_n = h \circ f$ for all $n$) is the desired function.

Case (2). $\text{Re} A$ is dense in $C_R(X)$ and $h = 0$ on $[-1, 0]$, so there exists a $u \in \text{Re} A$ such that $u(F_0) = 1$ and $u(F_1) = 0$. Since $\text{Re} A$ is ultraseparating, there exists a $G \in \ell^\infty(N, \text{Re} A)$ which separates $p$ and $q$. For each $n$ let $c_n$ denote $G(n, x_p)$. Then $g = (c_n u)$ is the desired function.

Case (3). Without loss of generality we may assume that $\tilde{f}(p) > 0$. Put $a = \sup\{\tilde{f}(x): x \in \beta(N \times F_1)\}$. Let $\tilde{w} = \tilde{f} - (a + 1)\tilde{u}$, where $u \in \text{Re} A$ is 0 on $F_0$ and 1 on $F_1$. Then $\tilde{w}(\beta(N \times F_1)) < 0$, $\tilde{w}(N \times \{x_p\}) = 0$, $\tilde{w}(p) > 0$ and $\tilde{w}(p) \neq \tilde{w}(q)$. Let $D = \{u \in \text{Re} A: u(x_p) = 0, u(F_1) \subset [-1, 0], -1 < u < 1\}$. For each $n$ let $D_n = \{u \in D: N(h \circ u) < n\}$. Then $D$ is closed in $\text{Re} A$ and $D = \bigcup_{n=1}^{\infty} D_n$. So by the Baire category theorem, the closure of some $D_n$ has nonempty interior in $D$. 

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Thus there are a \( u_0 \in D \), a positive integer \( r \), and an \( \epsilon > 0 \) such that \( U \cap D_r \) is dense in \( U \). Let \( U \neq \emptyset \) where \( U = \{ u \in \text{Re} A : N(u - u_0) < \epsilon \} \). We may assume that 
\[-1 < u_0 < 1 \text{ and } u_0(F_x) \subset (-1, 0). \]

Let \( W_x = \{ \tilde{u} = (u_n) \in l^\infty(N, \text{Re} A) : u_n \in D, \sup_n N(u_n - u) < \epsilon \} \). Then \( h \circ \tilde{u} \in \text{Cl}(l^\infty(N, \text{Re} A)) \) (the uniform closure of \( l^\infty(N, \text{Re} A) \) in \( C_R(\beta(N \times X)) \)) whenever \( \tilde{u} \in W_x \). For an appropriately small number \( t > 0 \), we have \( \tilde{u}_0 + t\tilde{w} \in W_x \). \( h \circ (\tilde{u}_0 + t\tilde{w})(p) = h \circ (t\tilde{w})(p) = h \circ (t\tilde{w})(q) \) and \(-1 < \tilde{u}_0 + t\tilde{w} < 0 \) on \( \beta(N \times F_1) \). There exists a sequence \( \{ v_k \} \) in \( l^\infty(N, \text{Re} A) \), where \( v^k = (v^k_n) \quad (v^k_n \in D_r \cap U) \), such that \( v^k \to \tilde{u}_0 + t\tilde{w} \) uniformly on \( \beta(N \times X) \) as \( k \to \infty \). For sufficiently large \( k \), we obtain that \( h \circ v^k(p) \neq h \circ v^k(q) \) and that \( h \circ v^k \) vanishes on \( \beta(N \times F_1) \). This function \( h \circ v^k \) is the desired function.

3. Proof of the theorem. Suppose that \( h \) operates weak-boundedly at \( x \) i.e., there exist an \( \epsilon > 0 \), a \( \delta > 0 \), a compact neighborhood \( F_x \) of \( x \) and a nonempty compact subset \( F_0 \) of \( X \) such that \( F_0 \cap F_x = \emptyset \) and \( h \circ u \in (\text{Re} A)_x \) and \( N_x(h \circ u) < \epsilon \) whenever \( u \in (\text{Re} A)_x^0 \) with \( N_x^0(u) < \delta \). Suppose that it follows that \( F_x \) is an interpolation set for \( A \). Then each point of \( X \), with finitely many exceptions, has a compact neighborhood which is an interpolation set for \( A \) by Lemma 1. Then we get \( A = C(X) \). So it is sufficient to prove that \( F_x \) is an interpolation set for \( A \).

Let \( V \) denote \( \text{Cl}(l^\infty(N, (\text{Re} A)_x)) \). We construct an algebra contained in \( V \) which contains the constants and separates the points of \( \beta(N \times F_x) \). Then the Stone-Weierstrass theorem will imply that \( V = C_R(\beta(N \times F_x)) \) and by Bernard’s lemma [1] \( (\text{Re} A)_x = C_R(F_x) \) so \( A|F_x = C(F_x) \) by the theorem of Sidney and Stout [6].

Let \( \tilde{u}_1 = (u_1^1), \tilde{u}_2 = (u_2^2), \tilde{u}_3 = (u_3^3), \ldots, \tilde{u}_m = (u_m^m) \) be in \( l^\infty(N, (\text{Re} A)_x^0) \). For sufficiently small \( \Delta \), let \( \lambda_\Delta \) be a nonnegative \( n \)-times continuously differentiable function supported in \( (-\Delta, \Delta) \) and with integral 1. Let \( \phi_\Delta \) denote the convolution
\[
\phi_\Delta(x) = \int_{-\Delta}^{\Delta} h(x - t)\lambda_\Delta(t)dt,
\]
where \( \phi_\Delta \) is \( n \)-times continuously differentiable and converges uniformly to \( h \) on any compact subinterval of \( (-1, 1) \) as \( \Delta \) tends to 0.

There exist an \( s_0 \in (-\delta/2, \delta/2) \) and a \( \Delta < \delta/2 \) such that \( \phi_\Delta^{(m)}(s_0) \neq 0 \). For if \( \phi_\Delta^{(m)} = 0 \) on \( (-\delta/2, \delta/2) \) for each small \( \gamma \), then \( \phi_\Delta \) is a polynomial of degree at most \( m - 1 \) so \( h \) is also a polynomial of degree at most \( m - 1 \) on \( (-\delta/2, \delta/2) \), which is a contradiction. For sufficiently small \( s_1, s_2, s_3, \ldots, s_m \), we have
\[
h \circ (s_0 + s_1u_1^1 + s_2u_2^2 + s_3u_3^3 + \cdots + s_mu_m^m - t) \in (\text{Re} A)_x
\]
and
\[
N_x(h \circ (s_0 + s_1u_1^1 + s_2u_2^2 + s_3u_3^3 + \cdots + s_mu_m^m - t)) < \epsilon
\]
for each \( n \) whenever \( |t| < \Delta \). Thus
\[
h \circ (s_0 + s_1\tilde{u}_1 + s_2\tilde{u}_2 + s_3\tilde{u}_3 + \cdots + s_m\tilde{u}_m - t) \in V
\]
if \( |t| < \Delta \), so
\[
\phi_\Delta \circ (s_0 + s_1\tilde{u}_1 + s_2\tilde{u}_2 + s_3\tilde{u}_3 + \cdots + s_m\tilde{u}_m) \in V.
\]
In particular fixing $s_2, s_3, \ldots, s_m$ and varying $s_1$ gives

$$\phi_\Delta \circ (s_0 + s_2\check{u}_2 + s_3\check{u}_3 + \cdots + s_m\check{u}_m) \in V$$

hence

$$\{\phi_\Delta \circ (s_0 + s_1\check{u}_1 + s_2\check{u}_2 + \cdots + s_m\check{u}_m)$$

$$-\phi_\Delta \circ (s_0 + s_2\check{u}_2 + \cdots + s_m\check{u}_m)\}/s_1 \in V$$

if $s_1$ is small and nonzero, and letting $s_1 \to 0$,

$$\phi_\Delta' \circ (s_0 + s_2\check{u}_2 + \cdots + s_m\check{u}_m)\check{u}_1 \in V$$

for small enough $s_2, s_3, \ldots, s_m$. Continuing in this manner, in $m$ stages we get

$$\phi_\Delta^{(m)}(s_0)\check{u}_1 \cdot \check{u}_2 \cdot \check{u}_3 \cdot \cdots \cdot \check{u}_m \in V.$$ 

Therefore the algebra generated by $l^\infty(N, (\Re A)^0)$ separates the points of $\beta(N \times F_x)$, hence this algebra is the desired algebra.

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REFERENCES