FUNCTIONS WHICH OPERATE ON THE REAL PART OF
A FUNCTION ALGEBRA

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Abstract. Recently S. J. Sidney [5] has shown that a "highly nonaffine" function \( h \) on an interval cannot operate by composition on the real part of a nontrivial function algebra. In this paper, we obtain the general result by considering the case in which \( h \) is not "highly nonaffine".

1. Introduction. Let \( A \) be a function algebra on a compact Hausdorff space \( X \) and \( h \) a nonaffine function on an interval \( I \). We say that \( h \) operates by composition on \( \text{Re} \ A \) if \( h \circ u \in \text{Re} \ A \) whenever \( u \in \text{Re} \ A \) has range in \( I \). We consider a conjecture: If \( h \) operates by composition on \( \text{Re} \ A \), then we have \( A = C(X) \). The theorem of J. Wermer [7] is equivalent to the conjecture for \( h(t) = t^2 \). A. Bernard [1] proved the conjecture for \( h(t) = |t| \). S. J. Sidney [5] obtained results for the cases that \( h \) is "highly nonaffine" or continuously differentiable. Our purpose is to show the following theorem.

Theorem. Suppose that \( A \) is a function algebra on a compact Hausdorff space \( X \) and that \( h \) is a nonaffine continuous function on an interval \( I \). If \( h \) operates by composition on \( \text{Re} \ A \), then we have \( A = C(X) \).

If \( h \) is nonaffine on every nondegenerate subinterval of \( I \), then \( h \) is "highly nonaffine". So without loss of generality we may assume that \( I = [-1, 1] \), \( h = 0 \) on \([-1, 0] \) and \( h \) is not affine on any open subinterval of \( I \) containing 0.

\( \text{Re} \ A \) is a Banach space with the usual quotient norm

\[
N(u) = \inf \{ \| f \| : f \in A, \text{Re} f = u \}.
\]

We denote \( f|F \) the restriction of a function \( f \in C(X) \) to a subset \( F \subset X \). For nonempty disjoint compact subsets \( F_1 \) and \( F_2 \) of \( X \) we denote

\[
(\text{Re} \ A)_1 = \{ u \in C_\text{R}(F_1) : \exists \hat{u} \in \text{Re} A, \hat{u}|F_1 = u \},
\]

\[
(\text{Re} \ A)_1^2 = \{ u \in C_\text{R}(F_1) : \exists \hat{u} \in \text{Re} A, \hat{u}|F_1 = u, \hat{u}|F_2 = 0 \}.
\]

For \( u \in (\text{Re} \ A)_1 \), we define

\[
N_1(u) = \inf \{ N(\hat{u}) : \hat{u} \in \text{Re} A, \hat{u}|F_1 = u \}.
\]

For \( u \in (\text{Re} \ A)_1^2 \), we define

\[
N_1^2(u) = \inf \{ N(\hat{u}) : \hat{u} \in \text{Re} A, \hat{u}|F_1 = u, \hat{u}|F_2 = 0 \}.
\]
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Then \((\text{Re } A)\)_1 and \((\text{Re } A)\)_2 are complete with respect to the norms \(N_1(\cdot)\) and \(N_2(\cdot)\), respectively.

2. Lemmas. Let \(x\) be a point of \(X\). We say that the function \(h\) operates weak-boundedly at \(x\) if there exist a \(\delta > 0\), an \(\varepsilon > 0\), a compact neighborhood \(F_x\) of \(x\), and a compact subset \(F_0\) of \(X\), which is disjoint from \(F_x\), with the following property: \(h \circ u \in (\text{Re } A)_x\) and \(N_x(h \circ u) < \varepsilon\) for each \(u \in (\text{Re } A)_x\) with \(N_x(u) < \delta\).

**Lemma 1.** Suppose that \(h\) operates by composition on \((\text{Re } A)_x\). Then \(h\) operates weak-boundedly at each point of \(X\) except for at most finitely many points.

**Proof.** Suppose that the lemma fails. Then there exists a countable subset \(\{x_n\}\) of \(X\) with the following two properties: (1) Each \(x_n\) has a compact neighborhood \(F_n\) such that \((\text{Cl}(\cup_{k\le n} F_k)) \cap F_n = \emptyset\) for each positive integer \(n\). (2) \(h\) does not operate weak-boundedly at each point \(x_n\). Let \(F_{m(n)}\) denote \(\text{Cl}(\cup_{k\le m(n)} F_k)\) for each \(n\) \((m(n)\) is the index depending on \(n)\). Since \(h\) does not operate weak-boundedly at \(x_n\), there exists a \(u_n \in (\text{Re } A)_{m(n)}\) such that \(N_{m(n)}(u_n) < 1/2^n\) and \(N_x(h \circ u_n) > n\) for each \(n\). There exists a \(\hat{u}_n \in \text{Re } A\) such that \(\hat{u}_n|F_n = u_n\), \(\hat{u}_n|F_{m(n)} = 0\) and \(N(\hat{u}_n) < 1/2^n\) for each \(n\). So \(\sum_{n=1}^{\infty} \hat{u}_n = \hat{u} \in \text{Re } A\) and \(h \circ \hat{u} \in \text{Re } A\) and \(\hat{u}|F_n = u_n\) for each \(n\). Thus \(N(h \circ \hat{u}) > N_x((h \circ \hat{u})|F_n) = N_n(u) > n\) for each \(n\). This contradicts \(h \circ \hat{u} \in \text{Re } A\).

**Lemma 2.** Let \(F_0\) and \(F_1\) be nonempty disjoint compact subsets of \(X\). Then \((\text{Re } A)_1\) is an ultraseparating Banach function space with respect to the norm \(N_1(\cdot)\).

**Proof.** For each \(p \in \beta(\text{N} \times F_0)\) the functional \(u \mapsto \tilde{u}(p)\) on \(\text{Cr}(F_0)\) is linear and multiplicative, so there is a unique \(x_p \in F_0\) such that \(\tilde{u}(p) = u(x_p)\) for all \(u \in \text{Cr}(X)\). \((\tilde{u} = (u_n)\) where \(u_n = u\) for all \(n)\). Let us take \(p, q \in \beta(\text{N} \times F_0)\) and \(p \neq q\). We shall find \(g \in l^\infty(\text{N}, \text{Re } A)\) such that \(g(x) = 0\) for \(x \in \beta(\text{N} \times F_1)\) and \(g(p) \neq g(q)\). We consider the following three cases:

1. \(x_p \neq x_q\).
2. \(x_p = x_q\), \(f(p) = f(q)\) whenever \(f \in l^\infty(\text{N}, \text{Re } A)\) vanishes on \(\text{N} \times \{x_p\}\).
3. \(x_p = x_q\), there exists an \(f \in l^\infty(\text{N}, \text{Re } A)\) such that \(f\) vanishes on \(\text{N} \times \{x_p\}\) and \(f(p) \neq f(q)\).

Case (1). Since \(\text{Re } A\) is uniformly dense in \(C_\text{R}(X)\), there exists an \(f \in \text{Re } A\) such that \(-1 < f(x_p) < 0\), \(f(F_1) \subset [-1, 0]\) and \(f(x_q) \not\in h^{-1}(0)\). Then \(g = h \circ f\) is the desired function.

Case (2). \(\text{Re } A\) is dense in \(C_\text{R}(X)\) and \(h = 0\) on \([-1, 0]\), so there exists a \(u \in \text{Re } A\) such that \(u(F_0) = 1\) and \(u(F_1) = 0\). Since \(\text{Re } A\) is ultraseparating, there exists a \(G \in l^\infty(\text{N}, \text{Re } A)\) which separates \(p\) and \(q\). For each \(n\) let \(c_n\) denote \(G(n, x_p)\). Then \(g = (c_n u)\) is the desired function.

Case (3). Without loss of generality we may assume that \(\tilde{f}(p) > 0\). Put \(\alpha = \sup\{\tilde{f}(\tilde{x})\}: \tilde{x} \in \beta(\text{N} \times F_1)\). Let \(\tilde{w} = \tilde{f} - (\alpha + 1) \tilde{u}\), where \(u \in \text{Re } A\) is 0 on \(F_0\) and 1 on \(F_1\). Then \(\tilde{w}(\beta(\text{N} \times F_1)) < 0\), \(\tilde{w}(\text{N} \times \{x_p\}) = 0\), \(\tilde{w}(p) > 0\) and \(\tilde{w}(q) \neq \tilde{w}(q)\). Let \(D = \{u \in \text{Re } A\}: u(x_1) = 0\), \(u(F_1) \subset [-1, 0]\), \(-1 < u < 1\). For each \(n\) let \(D_n = \{u \in D\}: N(h \circ u) < n\). Then \(D\) is closed in \(\text{Re } A\) and \(D = \bigcup_{n=1}^{\infty} D_n\). So by the Baire category theorem, the closure of some \(D_n\) has nonempty interior in \(D\).

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Thus there are a \( u_0 \in D \), a positive integer \( r \), and an \( \varepsilon > 0 \) such that \( U \cap D_r \) is dense in \( U \cap D \) where \( U = \{ u \in \text{Re} A : N(u - u_0) < \varepsilon \} \). We may assume that

\[-1 < u_0 < 1 \quad \text{and} \quad u_0(F_x) \subset (-1, 0).\]

Let \( W_\varepsilon = \{ \tilde{u} = (u_n) \in l^\infty(N, \text{Re} A) : u_n \in D, \sup_n N(u_n - u) < \varepsilon \} \). Then \( h \circ \tilde{u} \in \text{Cl}(l^\infty(N, \text{Re} A)) \) (the uniform closure of \( l^\infty(N, \text{Re} A) \) in \( C_R(\beta(N \times X)) \)) whenever \( \tilde{u} \in W_\varepsilon \). For an appropriately small number \( t > 0 \), we have \( \tilde{u}_0 + t\tilde{w} \in W_\varepsilon \), \( h \circ (\tilde{u}_0 + t\tilde{w})(p) = h \circ (t\tilde{w})(p) \neq h \circ (\tilde{w})(q) = h \circ (\tilde{u}_0 + t\tilde{w})(q) \) and \( -1 < \tilde{u}_0 + t\tilde{w} < 0 \) on \( \beta(N \times F_1) \). There exists a sequence \( \{ v^k \} \) in \( l^\infty(N, \text{Re} A) \), where \( v^k = (v^k_n) \ (v^k_n \in D_r \cap U) \), such that

\[ v^k \rightarrow \tilde{u}_0 + t\tilde{w} \quad \text{uniformly on} \ \beta(N \times X) \quad \text{as} \ k \rightarrow \infty. \]

For sufficiently large \( k \), we obtain that \( h \circ v^k(p) \neq h \circ v^k(q) \) and that \( h \circ v^k \) vanishes on \( \beta(N \times F_1) \). This function \( h \circ v^k \) is the desired function.

3. Proof of the theorem. Suppose that \( h \) operates weak-boundedly at \( x \) i.e., there exist an \( \varepsilon > 0 \), a \( \delta > 0 \), a compact neighborhood \( F_x \) of \( x \) and a nonempty compact subset \( F_0 \) of \( X \) such that \( F_0 \cap F_x = \emptyset \) and \( h \circ u \in (\text{Re} A)_x \) and \( N_x(h \circ u) < \varepsilon \) whenever \( u \in (\text{Re} A)_x \) with \( N^0_x(u) < \delta \). Suppose that it follows that \( F_x \) is an interpolation set for \( A \). Then each point of \( X \), with finitely many exceptions, has a compact neighborhood which is an interpolation set for \( A \) by Lemma 1. Then we get \( A = C(X) \). So it is sufficient to prove that \( F_x \) is an interpolation set for \( A \).

Let \( V \) denote \( \text{Cl}(l^\infty(N, (\text{Re} A)_x)) \). We construct an algebra contained in \( V \) which contains the constants and separates the points of \( \beta(N \times F_x) \). Then the Stone-Weierstrass theorem will imply that \( V = C_R(\beta(N \times F_x)) \), and by Bernard’s lemma [1] \( \text{Re} A \times = C_R(F_x) \) so \( A|_{F_x} = C(F_x) \) by the theorem of Sidney and Stout [6].

Let \( \tilde{u}_1 = (u^1_1), \tilde{u}_2 = (u^2_2), \tilde{u}_3 = (u^3_3), \ldots, \tilde{u}_m = (u^m_m) \) be in \( l^\infty(N, (\text{Re} A)_x^0) \). For sufficiently small \( \Delta \), let \( \lambda_\Delta \) be a nonnegative \( n \)-times continuously differentiable function supported in \( (-\Delta, \Delta) \) and with integral 1. Let \( \phi_\Delta \) denote the convolution

\[ \phi_\Delta(x) = \int_{-\Delta}^{\Delta} h(x - t)\lambda_\Delta(t)dt, \quad -1 + \Delta < x < 1 - \Delta. \]

\( \phi_\Delta \) is \( n \)-times continuously differentiable and converges uniformly to \( h \) on any compact subinterval of \( (-1, 1) \) as \( \Delta \) tends to 0.

There exist an \( s_0 \in (-\delta/2, \delta/2) \) and a \( \Delta < \delta/2 \) such that \( \phi_\Delta^{(m)}(s_0) \neq 0 \). For if \( \phi_\Delta^{(m)}(s_0) = 0 \) on \( (-\delta/2, \delta/2) \) for each small \( \gamma \), then \( \phi_\Delta \) is a polynomial of degree at most \( m - 1 \) so \( h \) is also a polynomial of degree at most \( m - 1 \) on \( (-\delta/2, \delta/2) \), which is a contradiction. For sufficiently small \( s_1, s_2, s_3, \ldots, s_m \) we have

\[ h \circ (s_0 + s_1u^1_n + s_2u^2_n + s_3u^3_n + \cdots + s_mu^m_n - t) \in (\text{Re} A)_x \]

and

\[ N_x(h \circ (s_0 + s_1u^1_n + s_2u^2_n + s_3u^3_n + \cdots + s_mu^m_n - t)) < \varepsilon \]

for each \( n \) whenever \( |t| < \Delta \). Thus

\[ h \circ (s_0 + s_1\tilde{u}_1 + s_2\tilde{u}_2 + s_3\tilde{u}_3 + \cdots + s_m\tilde{u}_m - t) \in V \]

if \( |t| < \Delta \), so

\[ \phi_\Delta \circ (s_0 + s_1\tilde{u}_1 + s_2\tilde{u}_2 + s_3\tilde{u}_3 + \cdots + s_m\tilde{u}_m) \in V. \]
In particular fixing $s_2, s_3, \ldots, s_m$ and varying $s_1$ gives

$$\phi_\Delta \circ (s_0 + s_2 \tilde{u}_2 + s_3 \tilde{u}_3 + \cdots + s_m \tilde{u}_m) \in \mathcal{V}$$

hence

$$\left\{ \phi_\Delta \circ (s_0 + s_1 \tilde{u}_1 + s_2 \tilde{u}_2 + \cdots + s_m \tilde{u}_m) \right\} / s_1 \in \mathcal{V}$$

if $s_1$ is small and nonzero, and letting $s_1 \to 0$,

$$\phi_\Delta \circ (s_0 + s_2 \tilde{u}_2 + \cdots + s_m \tilde{u}_m) \tilde{u}_1 \in \mathcal{V}$$

for small enough $s_2, s_3, \ldots, s_m$. Continuing in this manner, in $m$ stages we get

$$\phi_\Delta^{(m)}(s_0) \tilde{u}_1 \cdot \tilde{u}_2 \cdot \tilde{u}_3 \cdot \cdots \cdot \tilde{u}_m \in \mathcal{V}.$$ 

Therefore the algebra generated by $l^\infty(N, (\text{Re} A)^0)$ separates the points of $\beta(N \times F)$, hence this algebra is the desired algebra.

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REFERENCES


