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FUNCTIONS WHICH OPERATE ON THE REAL PART OF A FUNCTION ALGEBRA

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ABSTRACT. Recently S. J. Sidney [5] has shown that a "highly nonaffine" function h on an interval cannot operate by composition on the real part of a nontrivial function algebra. In this paper, we obtain the general result by considering the case in which h is not "highly nonaffine".

1. Introduction. Let A be a function algebra on a compact Hausdorff space X and h a nonaffinre function on an interval I. We say that h operates by composition on $\operatorname{Re} A$ if $h \circ u \in \operatorname{Re} A$ whenever $u \in \operatorname{Re} A$ has range in I. We consider a conjecture: If h operates by composition on $\operatorname{Re} A$, then we have $A = C(X)$. The theorem of J. Wermer [7] is equivalent to the conjecture for $h(t) = t^2$. A. Bernard [1] proved the conjecture for $h(t) = |t|$. S. J. Sidney [5] obtained results for the cases that h is "highly nonaffine" or continuously differentiable. Our purpose is to show the following theorem.

THEOREM. *Suppose that A is a function algebra on a compact Hausdorff space X and that h is a nonaffine continuous function on an interval I. If h operates by composition on* $\operatorname{Re} A$, *then we have $A = C(X)$.*

If h is nonaffine on every nondegenerate subinterval of I, then h is "highly nonaffine". So without loss of generality we may assume that $I = [-1, 1]$, $h = 0$ on $[-1, 0]$ and h is not affine on any open subinterval of I containing 0.

$\operatorname{Re} A$ is a Banach space with the usual quotient norm

$$N(u) = \inf\{\|f\|: f \in A, \operatorname{Re} f = u\}.$$

We denote $f|F$ the restriction of a function $f \in C(X)$ to a subset $F \subset X$. For nonempty disjoint compact subsets F_1 and F_2 of X we denote

$$(\operatorname{Re} A)_1 = \{u \in C_R(F_1): \exists \hat{u} \in \operatorname{Re} A, \hat{u}|F_1 = u\},$$

$$(\operatorname{Re} A)_1^2 = \{u \in C_R(F_1): \exists \hat{u} \in \operatorname{Re} A, \hat{u}|F_1 = u, \hat{u}|F_2 = 0\}.$$

For $u \in (\operatorname{Re} A)_1$, we define

$$N_1(u) = \inf\{N(\hat{u}): \hat{u} \in \operatorname{Re} A, \hat{u}|F_1 = u\}.$$

For $u \in (\operatorname{Re} A)_1^2$, we define

$$N_1^2(u) = \inf\{N(\hat{u}): \hat{u} \in \operatorname{Re} A, \hat{u}|F_1 = u, \hat{u}|F_2 = 0\}.$$

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Then $(\operatorname{Re} A)_1$ and $(\operatorname{Re} A)_1^2$ are complete with respect to the norms $N_1(\cdot)$ and $N_1^2(\cdot)$, respectively.

2. Lemmas. Let x be a point of X. We say that the function h *operates weak-boundedly at* x if there exist a $\delta > 0$, an $\varepsilon > 0$, a compact neighborhood F_x of x, and a compact subset F_0 of X, which is disjoint from F_x, with the following property: $h \circ u \in (\operatorname{Re} A)_x$ and $N_x(h \circ u) < \varepsilon$ for each $u \in (\operatorname{Re} A)_x^0$ with $N_x^0(u) < \delta$.

LEMMA 1. *Suppose that h operates by composition on* Re A. *Then h operates weak-boundedly at each point of X except for at most finitely many points.*

PROOF. Suppose that the lemma fails. Then there exists a countable subset $\{x_n\}$ of X with the following two properties: (1) Each x_n has a compact neighborhood F_n such that $(\operatorname{Cl}(\bigcup_{k \neq n} F_k)) \cap F_n = \varnothing$ for each positive integer n. (2) h does not operate weak-boundedly at each point x_n. Let $F_{m(n)}$ denote $\operatorname{Cl}(\bigcup_{k \neq n} F_k)$ for each n. ($m(n)$ is the index depending on n.) Since h does not operate weak-boundedly at x_n, there exists a $u_n \in (\operatorname{Re} A)_n^{m(n)}$ such that $N_n^{m(n)}(u_n) < 1/2^n$ and $N_n(h \circ u_n) > n$ for each n. There exists a $\hat{u}_n \in \operatorname{Re} A$ such that $\hat{u}_n | F_n = u_n$, $\hat{u}_n | F_{m(n)} = 0$ and $N(\hat{u}_n) < 1/2^n$ for each n. So $\sum_{n=1}^{\infty} \hat{u}_n = \hat{u} \in \operatorname{Re} A$ and $h \circ \hat{u} \in \operatorname{Re} A$ and $\hat{u} | F_n = u_n$ for each n. Thus $N(h \circ \hat{u}) \geqslant N_n((h \circ \hat{u}) | F_n) = N_n(h \circ u_n) > n$ for each n. This contradicts $h \circ \hat{u} \in \operatorname{Re} A$.

LEMMA 2. *Let F_0 and F_1 be nonempty disjoint compact subsets of X. Then* $(\operatorname{Re} A)_0^1$ *is an ultraseparating Banach function space with respect to the norm* $N_0^1(\cdot)$.

PROOF. For each $p \in \beta(N \times F_0)$ the functional $u \mapsto \bar{u}(p)$ on $C_R(F_0)$ is linear and multiplicative, so there is a unique $x_p \in F_0$ such that $\bar{u}(p) = u(x_p)$ for all $u \in C_R(X)$. ($\bar{u} = (u_n)$ where $u_n = u$ for all n.) Let us take $p, q \in \beta(N \times F_0)$ and $p \neq q$. We shall find $g \in l^{\infty}(N, \operatorname{Re} A)$ such that $g(\tilde{x}) = 0$ for $\tilde{x} \in \beta(N \times F_1)$ and $g(p) \neq g(q)$. We consider the following three cases:

(1) $x_p \neq x_q$.

(2) $x_p = x_q$, $\tilde{f}(p) = \tilde{f}(q)$ whenever $\tilde{f} \in l^{\infty}(N, \operatorname{Re} A)$ vanishes on $N \times \{x_p\}$.

(3) $x_p = x_q$, there exists an $\tilde{f} \in l^{\infty}(N, \operatorname{Re} A)$ such that $\tilde{f}$ vanishes on $N \times \{x_p\}$ and $\tilde{f}(p) \neq \tilde{f}(q)$.

Case (1). Since Re A is uniformly dense in $C_R(X)$, there exists an $f \in \operatorname{Re} A$ such that $-1 \leqslant f(x_p) \leqslant 0$, $f(F_1) \subset [-1, 0]$ and $f(x_q) \notin h^{-1}(0)$. Then $g = \overline{h \circ f}$ ($\overline{h \circ f} = (u_n)$ where $u_n = h \circ f$ for all n) is the desired function.

Case (2). Re A is dense in $C_R(X)$ and $h = 0$ on $[-1, 0]$, so there exists a $u \in \operatorname{Re} A$ such that $u(F_0) = 1$ and $u(F_1) = 0$. Since Re A is ultraseparating, there exists a $G \in l^{\infty}(N, \operatorname{Re} A)$ which separates p and q. For each n let c_n denote $G(n, x_p)$. Then $g = (c_n u)$ is the desired function.

Case (3). Without loss of generality we may assume that $\tilde{f}(p) > 0$. Put $\alpha = \sup\{\tilde{f}(\tilde{x}): \tilde{x} \in \beta(N \times F_1)\}$. Let $\tilde{w} = \tilde{f} - (\alpha + 1)\bar{u}$, where $u \in \operatorname{Re} A$ is 0 on F_0 and 1 on F_1. Then $\tilde{w}(\beta(N \times F_1)) < 0$, $\tilde{w}(N \times \{x_p\}) = 0$, $\tilde{w}(p) > 0$ and $\tilde{w}(p) \neq \tilde{w}(q)$. Let $D = \{u \in \operatorname{Re} A\colon\ u(x_p) = 0,\ u(F_1) \subset [-1, 0],\ -1 \leqslant u \leqslant 1\}$. For each n let $D_n = \{u \in D\colon N(h \circ u) < n\}$. Then D is closed in Re A and $D = \bigcup_{n=1}^{\infty} D_n$. So by the Baire category theorem, the closure of some D_n has nonempty interior in D.

Thus there are a $u_0 \in D$, a positive integer r, and an $\varepsilon > 0$ such that $U \cap D_r$ is dense in $U \cap D$ where $U = \{u \in \operatorname{Re} A\colon N(u - u_0) < \varepsilon\}$. We may assume that $-1 < u_0 < 1$ and $u_0(F_1) \subset (-1, 0)$. Let $W_\varepsilon = \{\tilde{u} = (u_n) \in l^\infty(N, \operatorname{Re} A)\colon u_n \in D, \sup_n N(u_n - u) < \varepsilon\}$. Then $h \circ \tilde{u} \in \operatorname{Cl}(l^\infty(N, \operatorname{Re} A))$ (the uniform closure of $l^\infty(N, \operatorname{Re} A)$ in $C_R(\beta(N \times X)))$ whenever $\tilde{u} \in W_\varepsilon$. For an appropriately small number $t > 0$, we have $\tilde{u}_0 + t\tilde{w} \in W_\varepsilon$, $h \circ (\tilde{u}_0 + t\tilde{w})(p) = h \circ (t\tilde{w})(p) \neq h \circ (t\tilde{w})(q) = h \circ (\tilde{u}_0 + t\tilde{w})(q)$ and $-1 < \tilde{u}_0 + t\tilde{w} < 0$ on $\beta(N \times F_1)$. There exists a sequence $\{v^k\}$ in $l^\infty(N, \operatorname{Re} A)$, where $v^k = (v_n^k)$ $(v_n^k \in D_r \cap U)$, such that $v^k \to \tilde{u}_0 + t\tilde{w}$ uniformly on $\beta(N \times X)$ as $k \to \infty$. For sufficiently large k, we obtain that $h \circ v^k(p) \neq h \circ v^k(q)$ and that $h \circ v^k$ vanishes on $\beta(N \times F_1)$. This function $h \circ v^k$ is the desired function.

3. Proof of the theorem. Suppose that h operates weak-boundedly at x i.e., there exist an $\varepsilon > 0$, a $\delta > 0$, a compact neighborhood F_x of x and a nonempty compact subset F_0 of X such that $F_0 \cap F_x = \varnothing$ and $h \circ u \in (\operatorname{Re} A)_x$ and $N_x(h \circ u) < \varepsilon$ whenever $u \in (\operatorname{Re} A)_x^0$ with $N_x^0(u) < \delta$. Suppose that it follows that F_x is an interpolation set for A. Then each point of X, with finitely many exceptions, has a compact neighborhood which is an interpolation set for A by Lemma 1. Then we get $A = C(X)$. So it is sufficient to prove that F_x is an interpolation set for A.

Let V denote $\operatorname{Cl}(l^\infty(N, (\operatorname{Re} A)_x))$. We construct an algebra contained in V which contains the constants and separates the points of $\beta(N \times F_x)$. Then the Stone-Weierstrass theorem will imply that $V = C_R(\beta(N \times F_x))$, and by Bernard's lemma [1] $(\operatorname{Re} A)_x = C_R(F_x)$ so $A|F_x = C(F_x)$ by the theorem of Sidney and Stout [6].

Let $\tilde{u}_1 = (u_n^1)$, $\tilde{u}_2 = (u_n^2)$, $\tilde{u}_3 = (u_n^3), \ldots, \tilde{u}_m = (u_n^m)$ be in $l^\infty(N, (\operatorname{Re} A)_x^0)$. For sufficiently small Δ, let λ_Δ be a nonnegative n-times continuously differentiable function supported in $(-\Delta, \Delta)$ and with integral 1. Let ϕ_Δ denote the convolution

$$\phi_\Delta(x) = \int_{-\Delta}^{\Delta} h(x - t)\lambda_\Delta(t)\,dt, \qquad -1 + \Delta < x < 1 - \Delta.$$

ϕ_Δ is n-times continuously differentiable and converges uniformly to h on any compact subinterval of $(-1, 1)$ as Δ tends to 0.

There exist an $s_0 \in (-\delta/2, \delta/2)$ and a $\Delta < \delta/2$ such that $\phi_\Delta^{(m)}(s_0) \neq 0$. For if $\phi_\gamma^{(m)} = 0$ on $(-\delta/2, \delta/2)$ for each small γ, then ϕ_γ is a polynomial of degree at most $m - 1$ so h is also a polynomial of degree at most $m - 1$ on $(-\delta/2, \delta/2)$, which is a contradiction. For sufficiently small $s_1, s_2, s_3, \ldots, s_m$ we have

$$h \circ \left(s_0 + s_1 u_n^1 + s_2 u_n^2 + s_3 u_n^3 + \cdots + s_m u_n^m - t\right) \in (\operatorname{Re} A)_x$$

and

$$N_x\left(h \circ \left(s_0 + s_1 u_n^1 + s_2 u_n^2 + s_3 u_n^3 + \cdots + s_m u_n^m - t\right)\right) < \varepsilon$$

for each n whenever $|t| < \Delta$. Thus

$$h \circ (s_0 + s_1\tilde{u}_1 + s_2\tilde{u}_2 + s_3\tilde{u}_3 + \cdots + s_m\tilde{u}_m - t) \in V$$

if $|t| < \Delta$, so

$$\phi_\Delta \circ (s_0 + s_1\tilde{u}_1 + s_2\tilde{u}_2 + s_3\tilde{u}_3 + \cdots + s_m\tilde{u}_m) \in V.$$

In particular fixing $s_2, s_3, \ldots, s_m$ and varying s_1 gives

$$\phi_\Delta \circ (s_0 + s_2\tilde{u}_2 + s_3\tilde{u}_3 + \cdots + s_m\tilde{u}_m) \in V$$

hence

$$\{\phi_\Delta \circ (s_0 + s_1\tilde{u}_1 + s_2\tilde{u}_2 + \cdots + s_m\tilde{u}_m) - \phi_\Delta \circ (s_0 + s_2\tilde{u}_2 + \cdots + s_m\tilde{u}_m)\}/s_1 \in V$$

if s_1 is small and nonzero, and letting $s_1 \to 0$,

$$\phi'_\Delta \circ (s_0 + s_2\tilde{u}_2 + \cdots + s_m\tilde{u}_m)\tilde{u}_1 \in V$$

for small enough $s_2, s_3, \ldots, s_m$. Continuing in this manner, in m stages we get

$$\phi_\Delta^{(m)}(s_0)\tilde{u}_1 \cdot \tilde{u}_2 \cdot \tilde{u}_3 \cdot \cdots \cdot \tilde{u}_m \in V.$$

Therefore the algebra generated by $l^\infty(N, (\operatorname{Re} A)_x^0)$ separates the points of $\beta(N \times F_x)$, hence this algebra is the desired algebra.

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