EXISTENCE OF SETS OF UNIQUENESS OF $l^p$
FOR GENERAL ORTHONORMAL SYSTEMS$^1$

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Abstract. It is proved that for every orthonormal complete system in $L^2(0, 1)$ there exists a set $A$, of measure arbitrarily close to 1, which carries no nonzero function with Fourier transform in $l^p$, for every $p < 2$.

1. Suppose $\{\phi_n\}_{n=1}^\infty$ is an orthonormal complete system (ONC) in $L^2(0, 1)$. We call a Lebesgue measurable set $E \subset (0, 1)$ a set of uniqueness of $l^p$ if no nonzero function $f \in L^2(0, 1)$, vanishing almost everywhere in the complement of $E$, satisfies the condition

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^p < +\infty,$$

where $\{\hat{f}(n)\}_{n=1}^\infty$ denotes the Fourier transform of $f$ with respect to the system $\{\phi_n\}$, i.e.

$$\hat{f}(n) = \int_0^1 f(x) \overline{\phi_n(x)} \, dx.$$

Y. Katznelson [6] first proved that the trigonometric system admits sets of uniqueness of $l^p$, for every $p < 2$, of Lebesgue measure arbitrarily close to 1 (see also [3]). Katznelson's theorem has been subsequently generalized to the system of characters of a nondiscrete locally compact abelian group by A. Figà-Talamanca and G. I. Gaudry [4], and to every uniformly bounded ONC by the author [2].

The aim of this paper is to prove a further extension of this result to every ONC. As a consequence we give a new proof of the generalization (due to W. Orlicz and A. M. Olevskii) of a well-known theorem of Carleman stating that there exists a continuous function $f$ such that

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^p = +\infty \quad \text{for every } p < 2.$$

2. For $1 < p < +\infty$ we use $\|f\|_p$ and $\|\hat{f}\|_p$ in their usual meanings. The following lemmas hold.

Received by the editors October 30, 1980.

1980 Mathematics Subject Classification. Primary 42C15, 42C25.

$^1$ This work was partially supported by the C.N.R.

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0002-9939/81/0000-0529/$02.00

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Lemma 1. Suppose $\phi_1, \ldots, \phi_N$ are functions in $L^2(0, 1)$, and $E$ is an interval contained in $(0, 1)$. If $\varepsilon > 0$ and $\delta > 0$, there exists a function $\Psi \in L^2(0, 1)$ such that:

(i) $\Psi(x) = 0$ if $x \notin E$;
(ii) $|\{x \in E/\Psi(x) \neq 1\}| < \delta|E|$;
(iii) $\|\Psi\|_2 < (2|E|/\delta)^{1/2}$;
(iv) $\left| \int \Psi(x) \phi_j(x) \, dx \right| < \varepsilon, j = 1, \ldots, N$.

Proof. Let $k = [1/\delta] + 1$ and let $n$ be a positive integer. We split $E$ in $k^n$ intervals $E_1, \ldots, E_{k^n}$ of the same measure. Set

$$
\psi_n(x) = \begin{cases} 1 & \text{if } x \in E \cup E_{k+1} \cup \cdots \cup E_{k^{n-1}+1}, \\ 0 & \text{if } x \in E \setminus E_1 \cup E_{k+1} \cup \cdots \cup E_{k^{n-1}+1}. 
\end{cases}
$$

A direct computation shows that $\psi_n$'s satisfy (i)–(iii) for every $n$; moreover, $(\psi_n)_{n=1}^\infty$ tends to 0 weakly, and putting $\Psi = \psi_n$, with $n$ large enough, (iv) is satisfied too.

Lemma 2. Suppose $(\phi_n)$ is an ONC. Then, for every $\delta > 0$, $q > 2$, $0 < a < 1$, there exists a function $\Psi \in L^2(0, 1)$ such that:

(i) $\Psi(x) = 0$ if $x \notin (0, a)$;
(ii) $|\{x \in (0, a)/\Psi(x) \neq 1\}| < \delta$;
(iii) $\|\Psi\|_q < \delta$.

Proof. Let $\varepsilon$ and $\eta$ be positive numbers to be specified later. Divide $(0, a)$ into $m$ intervals $E_1, \ldots, E_m$ of measure less than $\eta$. We shall define the required function $\Psi$ piecewise on every $E_i$.

Let

$$
\psi_1(x) = \begin{cases} 1 & \text{if } x \in E_1, \\ 0 & \text{if } x \notin E_1,
\end{cases}
$$

and put $n_1 = 1$.

Suppose now $\psi_1, \ldots, \psi_{i-1}$ have already been defined. Then there exists an integer $n_i$ such that

$$
\left| \sum_{j=1}^{i-1} \psi_j(n) \right| < \varepsilon \quad \text{for every } n > n_i,
$$

and, via Lemma 1, it is possible to construct a function $\psi_i \in L^2(0, 1)$ such that:

$\psi_i(x) = 0$ if $x \notin E_i$;

$n|\{x \in E_i/\psi_i(x) \neq 1\}| < \delta|E_i|$;

$\|\psi_i\|_2 < (2|E_i|/\delta)^{1/2}$, and

$$
\left| \hat{\psi_i}(n) \right| < \varepsilon/2^i \quad \text{for every } n < n_i.
$$

Put $\Psi = \sum_{i=1}^n \psi_i$. It is easy to see that $\Psi$ satisfies (i) and (ii). Moreover,

$$
\|\Psi\|_2 < (2/\delta)^{1/2}.
$$
In order to prove (iii) we observe that for every \( i \) and every \( n \),
\[
|\hat{\psi}_i(n)| < \|\psi_i\|_2 < (2|E_i|/\delta)^{1/2} < (\delta^{1/2})^{1/2} < \epsilon
\]
if \( \eta = \eta(\epsilon) \) is chosen small enough.

Then, if \( n > n_m \), from (1) and (4) it follows that
\[
|\hat{\Psi}(n)| < \left| \sum_{j=1}^{m-1} \hat{\psi}_j(n) \right| + |\hat{\psi}_m(n)| < 2\epsilon,
\]
and, if \( n_{i-1} < n < n_i \),
\[
|\hat{\Psi}(n)| < \left| \sum_{j=1}^{i-2} \hat{\psi}_j(n) \right| + |\hat{\psi}_{i-1}(n)| + \sum_{j=i}^{m} |\hat{\psi}_j(n)| = I_1 + I_2 + I_3.
\]

But, it follows from (1) that \( I_1 < \epsilon \), from (4) that \( I_2 < \epsilon \), and from (2) that \( I_3 < \sum_{j=1}^{m} (\epsilon/2^j) < \epsilon \). Collecting these results we obtain
\[
(5) \quad \|\hat{\Psi}\|_\infty < 3\epsilon,
\]
and so, from (3) and (5),
\[
\|\hat{\Psi}\|_q < \|\hat{\Psi}\|_2^{2/q} \cdot \|\hat{\Psi}\|_\infty^{(q-2)/q} < (2/\delta)^{1/q} \cdot (3\epsilon)^{(q-2)/q} < \delta
\]
if \( \epsilon \) is small enough. \( \square \)

Remark 1. This lemma was originally proved by Y. Katznelson for the trigonometric system, and subsequently extended to any uniformly bounded ONC by A. Figà-Talamanca and G. I. Gaudry. Our proof, which holds for any, possibly unbounded, ONC is based on an idea of G. Alexits (see [1, Chapter II, §11]).

Theorem. For every ONC \( \{\phi_n\} \), and every \( \epsilon > 0 \), there exists a measurable set \( A \subset (0, 1) \), with \( |A| < \epsilon \), such that if \( f \in L^2(0, 1) \) vanishes a.e. in \( A \), and \( \|f\|_p < +\infty \) for some \( p < 2 \), then \( f(x) = 0 \) a.e. in \( (0, 1) \).

The theorem follows from Lemma 2 as in [2].

Remark 2. I. I. Hirschman and Y. Katznelson [5] proved that the trigonometric system admits closed sets which are sets of uniqueness of \( l^p \), but not of \( l^p \), with \( p < p' < 2 \). For an arbitrary ONC this feature fails to hold, as is shown in [2].

3. It is interesting to notice that, using our theorem, it is possible to prove easily the extension of a well-known theorem of T. Carleman to every ONC (see [7, Chapter III, §4] for the original proof of this extension).

Theorem. For every ONC \( \{\phi_n\} \) there exists a continuous bounded function \( f \) such that \( \|f\|_p = +\infty \) for every \( p < 2 \).

Proof. See [2].

Remark 3. The theorems stated for orthonormal systems in \( L^2(0, 1) \) can be easily extended to orthonormal systems in \( L^2(-\infty, +\infty) \) or to orthonormal systems of square integrable functions over more general measure spaces.
REFERENCES


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