CURVATURE ESTIMATES FOR COMPLETE AND BOUNDED SUBMANIFOLDS IN A RIEMANNIAN MANIFOLD

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ABSTRACT. Let $\mathcal{M}$ be a complete $n$-dimensional submanifold in the $(2n-1)$-dimensional Euclidean space, with scalar curvature bounded from below. Baikousis and Koufogiorgos proved that the sectional curvature of $\mathcal{M}$ satisfies $\sup K_M > \lambda^{-2}$ if $\mathcal{M}$ is contained in a ball of radius $\lambda$. We extend this result to the case that the ambient space is a complete simply connected Riemannian manifold of nonpositive curvature.

1. Introduction. For $p < n$, let $\mathcal{M}$ be a complete $n$-dimensional Riemannian submanifold in the $(n+p)$-dimensional Euclidean space $E^{n+p}$. Under the assumption that the scalar curvature of $\mathcal{M}$ has a lower bound, Baikousis and Koufogiorgos [1] proved that if $\mathcal{M}$ is contained in a ball of radius $\lambda$, then the sectional curvature $K_M$ of $\mathcal{M}$ satisfies $\sup K_M > \lambda^{-2}$. In this note we obtain a natural extension of the above inequality when the ambient space is a complete simply connected $(n+p)$-dimensional Riemannian manifold of nonpositive curvature. To state our result, we introduce a continuous function $f$: $[0, \infty) \to [1, \infty)$ by

$$f(t) = \begin{cases} 1 & \text{if } t = 0, \\ t \coth(t) & \text{if } t > 0. \end{cases}$$

THEOREM. For $p < n$, let $\mathcal{M}$ be a complete $n$-dimensional Riemannian submanifold in a $(n+p)$-dimensional complete simply connected Riemannian manifold $\overline{\mathcal{M}}$ whose sectional curvature satisfies $a < K_{\overline{\mathcal{M}}} < b < 0$. If $\mathcal{M}$ is contained in a geodesic ball of radius $\lambda$ and the scalar curvature of $\mathcal{M}$ has a lower bound, then the sectional curvature $K_M$ of $\mathcal{M}$ satisfies $\sup K_M > a + \lambda^{-2}\{f(\sqrt{-b\lambda})\}^2$.

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2. Proof of Theorem. We denote the Riemannian metric on $\overline{\mathcal{M}}$ (resp. $\mathcal{M}$) by $\langle \cdot, \cdot \rangle$ (resp. $\langle \cdot, \cdot \rangle$), the Riemannian connection by $\nabla$ (resp. $\nabla$), the Riemannian curvature tensor by $\mathcal{R}$ (resp. $\mathcal{R}$) and the second fundamental form with respect to the immersion $\mathcal{M} \subset \overline{\mathcal{M}}$ by $a$.

Since the scalar curvature of $\mathcal{M}$ has a lower bound, we may assume $\inf K_M > -\infty$. Let $d$ be the distance function on $\overline{\mathcal{M}}$ and choose a point $\bar{\mathcal{O}} \in \overline{\mathcal{M}}$ such that $d(\bar{\mathcal{O}}, x) < \lambda$ for all $x \in \mathcal{M}$. We define a smooth function $F$: $\mathcal{M} \to R$ by $F(x) = \{d(\bar{\mathcal{O}}, x)\}^2/2$. Then by [4, Theorem A'] there exists a sequence $(x_k)_{k=1}^\infty$ in $\mathcal{M}$ such that $\|\text{grad } F(x_k)\| < k^{-1}$. 

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\[ \nabla^2 F(X, X) < k^{-1} \] for all unit vectors \( X \in T_xM \),

\[ \lim_{k \to \infty} F(x_k) = \sup F, \]

where \( \nabla^2 F \) denotes the Hessian of \( F \) with respect to the Riemannian metric on \( M \).

**Lemma 1.** Let \( \gamma: [0, 1] \to \bar{M} \) be a geodesic in \( \bar{M} \) such that \( \gamma(0) = \bar{o} \) and \( \gamma(1) \in M \). Then

\[ \nabla^2 F(X, X) > \langle \alpha(X, X), \dot{\gamma}(1) \rangle + L^{-2} \langle X, \dot{\gamma}(1) \rangle^2 \]

\[ + \left( \|X\|^2 - L^{-2} \langle X, \dot{\gamma}(1) \rangle^2 \right) f(\sqrt{-b} L), \]

for all vectors \( X \) tangent to \( M \) at \( \gamma(1) \), where \( L \) is the length of \( \gamma \).

**Proof.** Let \( c(s) \) be the geodesic in \( M \) such that \( c(0) = X \) and let \( \gamma_s: [0, 1] \to \bar{M} \) be the geodesic such that \( \gamma_s(0) = \bar{o} \) and \( \gamma_s(1) = c(s) \). Then we have \( \nabla^2 F(X, X) = F(c(s))\vert_{s=0}' = E(\gamma_s)\vert_{s=0}' \), where \( E(\gamma_s) \) is the energy of \( \gamma_s \) defined by \( E(\gamma_s) = \int_0^1 \langle \dot{\gamma}_s, \dot{\gamma}_s \rangle/2 \). Let \( V \) be the variation vector field along \( \gamma \) with respect to the variation \( \{\gamma_s\} \). Then a calculation shows that

\[ E(\gamma_s)'\vert_{s=0} = \langle \alpha(X, X), \dot{\gamma}(1) \rangle + I(V, V), \]

where \( I(V, V) = \int_0^1 \left( \langle \nabla^2 \gamma V, \nabla^2 \gamma V \rangle + \langle \nabla^2 \dot{\gamma}, \nabla^2 \dot{\gamma} \rangle \right) \). Let \( \bar{M} \) be the \((n + p)\)-dimensional space form with constant curvature \( b \) and let \( \sigma: [0, 1] \to \bar{M} \) be a geodesic with length \( L \). We construct a vector field \( W \) along \( \sigma \) such that \( \|V\| = \|\nabla^2 \gamma W\| = \|\nabla^2 \dot{\gamma} W\| \) and \( \langle V, \dot{\gamma} \rangle = \langle W, \dot{\gamma} \rangle \), where \( \nabla \) is the Riemannian connection with respect to the Riemannian metric \( \langle \cdot, \cdot \rangle \) on \( \bar{M} \). Then \( K_{\bar{M}} < b \) implies \( I(V, V) > I(W, W) \). Let \( J \) be the Jacobi field along \( \sigma \) determined by \( J(0) = 0 \) and \( J(1) = W(1) \). Then [2, First lemma, p. 24] implies \( I(W, W) > I(J, J) \). Let \( U \) be the parallel vector field along \( \sigma \) determined by \( U(1) = J(1) - L^{-2} \langle J(1), \dot{\gamma}(1) \rangle \dot{\gamma}(1) \), and let \( g: [0, 1] \to R \) be the solution of \( g'' + bL^2g = 0 \) determined by \( g(0) = 0 \) and \( g(1) = 1 \). Then we have \( J(t) = g(t)U(t) + \{L^{-2}J(1), \dot{\gamma}(1) \dot{\gamma}(1) \} \dot{\gamma}(1) \) and \( g'(1) = f(\sqrt{-b} L) \). Hence we see that \( I(J, J) = \langle \nabla^2 \dot{\gamma} J, J \rangle\vert_{s=1} = g'(1)\|U(1)\|^2 + L^{-2} \langle J(1), \dot{\gamma}(1) \rangle^2 = f(\sqrt{-b} L)\|X\|^2 - L^{-2} \langle X, \dot{\gamma}(1) \rangle^2 + \sqrt{-b} \langle X, \dot{\gamma}(1) \rangle^2 \). Q.E.D.

Let \( \gamma_k: [0, 1] \to \bar{M} \) be the geodesic such that \( \gamma_k(0) = \bar{o} \) and \( \gamma_k(1) = x_k \), and let \( \lambda_k \) be the length of \( \gamma_k \). We set \( \lambda_\infty = \sup \{d(\bar{o}, x) | x \in M \} \), then (4) implies \( \lim_{k \to \infty} \lambda_k = \lambda_\infty > 0 \). Therefore we may assume \( \lambda_k > 0 \) for all \( k \). Let \( X \) be a unit vector in \( T_xM \). Then by (3) and Lemma 1 we have

\[ k^{-1} \langle \alpha(X, X), \dot{\gamma}_k(1) \rangle - \lambda_k^2 \langle X, \dot{\gamma}_k(1) \rangle^2 \{ f(\sqrt{-b} \lambda_k) - 1 \} + f(\sqrt{-b} \lambda_k). \]

Since \( \langle X, \dot{\gamma}_k(1) \rangle = \langle X, \text{grad} F(x_k) \rangle \), (2) implies \( \langle X, \dot{\gamma}_k(1) \rangle^2 < k^{-2} \). Hence we have

\[ \| \alpha(X, X) \| \geq \left\{ f(\sqrt{-b} \lambda_k) - A_k \right\} / \lambda_k \]

for all unit vectors \( X \in T_xM \), where \( A_k = k^{-1} + k^{-2} \lambda_k^2 \{ f(\sqrt{-b} \lambda_k) - 1 \} \). Since \( \lim_{k \to \infty} \{ f(\sqrt{-b} \lambda_k) - A_k \} = f(\sqrt{-b} \lambda_\infty) > 1 \), we may assume \( f(\sqrt{-b} \lambda_k) - A_k > 0 \) for all \( k \). Hence (5) implies \( \alpha(X, X) \neq 0 \) for all nonzero vectors \( X \in T_xM \). Now we recall the following lemma [3, p. 28].

**Lemma 2.** Let \( \alpha: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \) be symmetric bilinear and satisfy \( \alpha(X, X) \neq 0 \) for all nonzero \( X \in \mathbb{R}^n \). If \( p < n \), there exist linearly independent vectors \( X, Y \in \mathbb{R}^n \) such that \( \alpha(X, Y) = 0, \alpha(X, X) = \alpha(Y, Y) \).
By Lemma 2 there exist linearly independent vectors $X_k, Y_k$ in $T_{x_k}M$ such that $\alpha(X_k, Y_k) = 0$, $\alpha(X_k, X_k) = \alpha(Y_k, Y_k)$. Hence by the Gauss equation, we have

$$\langle R(X_k, Y_k)Y_k, X_k \rangle = \langle R(X_k, Y_k)Y_k, X_k \rangle + \|\alpha(X_k, X_k)\| \cdot \|\alpha(Y_k, Y_k)\|.$$  

Let $K(X_k, Y_k)$ (resp. $K(X_k, Y_k)$) be the sectional curvature of $M$ (resp. $M$) for the plane spanned by $X_k$ and $Y_k$. Then by (5) we see that

$$K(X_k, Y_k) = \overline{K}(X_k, Y_k) + \|\alpha(X_k, X_k)\| \cdot \|\alpha(Y_k, Y_k)\| \cdot \|X_k\|^2 \|Y_k\|^2$$

$$> a + \|\alpha(X_k, X_k)\| \cdot \|\alpha(Y_k, Y_k)\| \cdot \|X_k\|^2 \|Y_k\|^2$$

$$> a + \lambda_k^{-2}\{f(\sqrt{-b} \lambda_k) - A_k\}^2.$$  

Letting $k$ go to infinity, we have $\sup K_M > a + \lambda^{-2}\{f(\sqrt{-b} \lambda)\}^2$. Since $\lambda_{\infty} < \lambda$ and the function $t \mapsto t^{-2}\{f(\sqrt{-b} t)\}^2$ is decreasing, we have $\sup K_M > a + \lambda^{-2}\{f(\sqrt{-b} \lambda)\}^2$. This completes the proof of the theorem.

REFERENCES


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