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A SUFFICIENT CONDITION FOR LINEAR GROWTH OF VARIANCES IN A STATIONARY RANDOM SEQUENCE

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ABSTRACT. Suppose $(X_k, k = \ldots, -1, 0, 1, \ldots)$ is a weakly stationary random sequence. For each positive integer n let $S_n \equiv X_1 + \cdots + X_n$ and let $\tau(n) = \operatorname{Sup}\{|\operatorname{Corr}(\Sigma_{k=-I}^{0} X_k, \Sigma_{k=m}^{m+I} X_k)|: m > n, I > 0\}$. If $\operatorname{Var} S_n \to \infty$ as $n \to \infty$ and $\Sigma_{n=0}^{\infty} \tau(2^n) < \infty$, then $n^{-1} \operatorname{Var} S_n$ converges to a finite positive limit as $n \to \infty$. A bound on the rate of convergence is obtained.

Let $(X_k, k = \ldots, -1, 0, 1, \ldots)$ be a weakly stationary sequence of random variables on a probability space $(\Omega, \mathfrak{F}, P)$. That is, $EX_k^2 < \infty \ \forall k$, there are constants μ and $\gamma(0)$ such that $EX_k = \mu$ and $\operatorname{Var} X_k = \gamma(0) \ \forall k$, and there is a sequence of numbers $\gamma(1), \gamma(2), \gamma(3), \ldots$ such that $\operatorname{Cov}(X_k, X_l) = \gamma(|k - l|)$ $\forall k \neq l$. For $-\infty \leqslant J \leqslant L \leqslant \infty$ let H_J^L denote the L^2-closure of the linear space spanned by the r.v.'s $X_k - \mu$, $J \leqslant k \leqslant L$. For $n = 1, 2, 3, \ldots$ let $S_n = X_1 + X_2 + \cdots + X_n$ and define the quantities

$$r(n) = \operatorname{Sup}\{|\operatorname{Corr}(f, g)|: f \in H_{-\infty}^{0}, g \in H_n^{\infty}\},$$

$$\tau(n) = \operatorname*{Sup}_{\substack{m > n \\ I > 0}} \left| \operatorname{Corr}\left(\sum_{k=-I}^{0} X_k, \sum_{k=m}^{m+I} X_k \right) \right|.$$

Obviously the sequences $\{r(n)\}$ and $\{\tau(n)\}$ are each nonincreasing, and $0 \leqslant \tau(n) \leqslant r(n)$.

Ibragimov and Rozanov [3, Note 2, p. 190] proved the following theorem (it was originally proved in [4]).

THEOREM 0 (IBRAGIMOV AND ROZANOV). *If (X_k) is weakly stationary and $\Sigma_{n=0}^{\infty} r(2^n) < \infty$, then (X_k) has an absolutely continuous spectral distribution function, with a continuous spectral density $f(\lambda)$.*

In [3, Note 2, p. 190] this is stated for stationary Gaussian sequences, but the extension to general weakly stationary sequences is easy. Ibragimov [2, Theorem 2.2] proved a central limit theorem for *strictly* stationary random sequences satisfying a condition similar to but stronger than $\Sigma\, r(2^n) < \infty$; in that theorem it was assumed that $f(0) > 0$. In central limit theorems for weakly dependent random variables it is common practice to assume $\operatorname{Var} S_n \to \infty$ as $n \to \infty$. Consider the question whether $f(0) > 0$ follows if one assumes $\Sigma\, r(2^n) < \infty$ and $\operatorname{Var} S_n \to \infty$.

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The answer is affirmative, and one can prove this as a corollary to Theorem 0 by using the results in Chapters 4 and 5 of [3]. It also follows easily from the following theorem.

THEOREM 1. *If* (X_k) *is weakly stationary*, $\operatorname{Var} S_n \to \infty$ *as* $n \to \infty$, *and* $\sum_{n=0}^{\infty} \tau(2^n) < \infty$, *then the following statements hold*:

(i) *There exists a finite positive constant* σ^2 *such that* $\operatorname{Lim}_{n\to\infty} n^{-1} \operatorname{Var} S_n = \sigma^2$.

(ii) *For each* γ, $0 < \gamma < 1$, *there exist positive constants* C *and* D *such that for all* $m \geqslant 1$,

$$|\sigma^2 - m^{-1} \operatorname{Var} S_m| \leqslant C\left(m^{(\gamma-1)/2} + \sum_{n=0}^{\infty} \tau\left(\left[(2^n m)^\gamma / D\right] + 1\right)\right)$$

where $[t]$ *denotes the greatest integer* $\leqslant t$.

Theorem 1 covers the case where $\tau(n) = o((\log n)^{-(1+\varepsilon)})$ for some $\varepsilon > 0$. For the stationary Gaussian sequence given in Example 1 on pp. 179–180 of [3], $r(n) = O((\log n)^{-1})$ and $n^{-1} \operatorname{Var} S_n \to \infty$ as $n \to \infty$. Berkes and Philipp [1, Theorem 4] prove an "almost sure invariance principle" for strictly stationary random sequences satisfying the ϕ-mixing condition with a rather slow (logarithmic) mixing rate. Their conditions satisfy the hypothesis of Theorem 1 (see [1, Lemma 4.1.1, p. 44]), and it would be interesting to see if one can choose $a_N = \sigma^2 N$ in the conclusion of their theorem. In the cases where $\tau(n) \to 0$ rapidly, other methods generally give much better bounds on $|\sigma^2 - m^{-1} \operatorname{Var} S_m|$ than Theorem 1(ii) does.

PROOF OF THEOREM 1. We will assume $EX_k = 0$ and $\operatorname{Var} X_k = 1$. Let $S_0 \equiv 0$. For any r.v. X let $\|x\| \equiv (EX^2)^{1/2}$. For each $n = 1, 2, 3, \ldots$ let $g(n) \equiv n^{-1/2}\|S_n\|$.

LEMMA 1. *If* $N \geqslant 1$, $K \geqslant 0$, *and* $\tau(K+1) < \frac{1}{2}$, *then for* $L = 2N$ *and for* $L = 2N + 1$ *one has*

$$\begin{aligned}\|S_N\| \cdot \left[2(1 - \tau(K+1))\right]^{1/2}(1 - 2b/\|S_N\|) &\leqslant \|S_L\| \\ &\leqslant \|S_N\| \cdot \left[2(1 + \tau(K+1))\right]^{1/2}(1 + 2b/\|S_N\|)\end{aligned}$$

where $b \equiv \max\{\|S_{K-1}\|, \|S_K\|\}$.

PROOF.

$$S_L = S_N + (S_{N+K} - S_N) + (S_{2N+K} - S_{N+K}) - (S_{2N+K} - S_L),$$

$$\begin{aligned}2(1 - \tau(K+1))\|S_N\|^2 &\leqslant \|S_N + S_{2N+K} - S_{N+K}\|^2 \\ &\leqslant 2(1 + \tau(K+1))\|S_N\|^2\end{aligned} \tag{1}$$

and Lemma 1 follows from Minkowski's inequality.

LEMMA 2. *Given any* $\varepsilon > 0$ *and any positive integer* L_0, *there are positive integers* N *and* L *with* $L \geqslant L_0$ *such that* $\forall\, l$, $L \leqslant l \leqslant 2L$, *one has*

$$(1 - \varepsilon)g(N) \leqslant g(l) \leqslant (1 + \varepsilon)g(N).$$

PROOF. For each integer $J \geqslant 2$, $\operatorname{Corr}(S_N, S_{JN} - S_{(J-1)N}) \to 0$ as $N \to \infty$. (For $J = 2$ this just follows from (1).) Hence $\forall\, J \geqslant 2$, $g(NJ)/g(N) \to 1$ as $N \to \infty$.

Let M be a positive integer such that $(1-\varepsilon) < (1-\varepsilon)^{1/2} - M^{-1/2} < (1+\varepsilon)^{1/2} + M^{-1/2} < (1+\varepsilon)$. Let N be such that $MN \geqslant L_0$,

$$\|S_N\| = \max\{\|S_k\|, 1 \leqslant k \leqslant N\},$$

and for each m, $M \leqslant m \leqslant 2M$, one has

$$(1-\varepsilon)^{1/2} < g(mN)/g(N) \leqslant (1+\varepsilon)^{1/2}.$$

Let $L = MN$.

If $L \leqslant l \leqslant 2L$, then for some m one has $M \leqslant m \leqslant 2M-1$ and $mN \leqslant l \leqslant (m+1)N$, and hence

$$\begin{aligned}(1-\varepsilon)g(N) &< \left[(1-\varepsilon)^{1/2} - (m+1)^{-1/2}\right] g(N)\\ &\leqslant \left[(m+1)N\right]^{-1/2}(\|S_{(m+1)N}\| - \|S_N\|)\\ &\leqslant g(l) \leqslant (mN)^{-1/2}(\|S_{mN}\| + \|S_N\|)\\ &\leqslant \left[(1+\varepsilon)^{1/2} + m^{-1/2}\right] g(N) < (1+\varepsilon)g(N)\end{aligned}$$

and Lemma 2 is proved.

Let $0 < A < 1$ be sufficiently small that if $(a_n, n = 1, 2, \ldots)$ is any sequence of real numbers such that $\Sigma_n|a_n| \leqslant A$ then $|1 - \Pi_n(1+a_n)| \leqslant 2\Sigma_n|a_n|$ and $|1 - \Pi_n(1+a_n)^{-1}| \leqslant 2\,\Sigma_n|a_n|$. Let [ ] denote the greatest-integer function.

Proof of Theorem 1(i). Assume $0 < \varepsilon < A$. Let N be a positive integer such that $\Sigma_{n=0}^{\infty} \tau([2^{N+n/6}]) < \varepsilon/6$. Let $C > 0$ be such that $\Sigma_{n=0}^{\infty} 2^{N+1-n/6}C^{-1} < \varepsilon/6$. Let L_0 be a positive integer such that (i) $\forall\, J \geqslant L_0$, $\|S_J\| \geqslant C$, (ii) $\Sigma_{n=0}^{\infty} 2^{-n}L_0^{-1} < \varepsilon/6$, and (iii) $\forall\, J \geqslant L_0$, $g(2J) > 2^{-1/6}g(J)$ and $g(2J+1) > 2^{-1/6}g(J)$ (see Lemma 1). For each $n = 0, 1, 2, \ldots$ let $K_n = [2^{N+n/6}]$. Using Lemma 2, let the positive integers H and L be such that $L \geqslant L_0$ and $\forall\, l$, $L \leqslant l \leqslant 2L$, one has $(1-\varepsilon)g(H) < g(l) < (1+\varepsilon)g(H)$.

Let m be an arbitrary positive integer satisfying $m \geqslant 2L$. We wish to prove $(1-\varepsilon)^2 g(H) < g(m) < (1+\varepsilon)^2 g(H)$.

For some positive integer M, $2^M L \leqslant m \leqslant 2^{M+1}L$. There is a sequence of positive integers $J_0, J_1, \ldots, J_M$ such that $m = J_M$, $L \leqslant J_0 \leqslant 2L$, and for each $n = 0, 1, \ldots, M-1$, $J_{n+1} = 2J_n$ or $J_{n+1} = 2J_n + 1$.

For each $n = 0, 1, \ldots, M-1$, one has $\|S_{J(n)}\| \geqslant 2^{n/3}\|S_{J(0)}\| \geqslant 2^{n/3}C$, and using Lemma 1 and the inequality $\|S_K\| \leqslant K$ one has

$$\begin{aligned}g(J_{n+1}) &\geqslant g(J_n)(2J_n/J_{n+1})^{1/2}(1 - \tau(K_n+1))^{1/2}(1 - 2K_n/(2^{n/3}C))\\ &\geqslant g(J_n)(1 - 2^{-n}L_0^{-1})^{1/2}(1 - \tau(K_n+1))^{1/2}(1 - 2^{N+1-n/6}C^{-1}),\\ g(J_{n+1}) &\leqslant g(J_n)(1 + \tau(K_n+1))^{1/2}(1 + 2K_n/(2^{n/3}C))\\ &\leqslant g(J_n)(1 + \tau(K_n+1))^{1/2}(1 + 2^{N+1-n/6}C^{-1}).\end{aligned}$$

Hence

$$g(m) \geqslant g(J_0) \prod_{n=0}^{M-1} \left[(1 - 2^{-n}L_0^{-1})^{1/2}(1 - \tau(K_n+1))^{1/2}(1 - 2^{N+1-n/6}C^{-1})\right],$$

$$g(m) \leqslant g(J_0) \prod_{n=0}^{M-1} \left[(1 + \tau(K_n+1))^{1/2}(1 + 2^{N+1-n/6}C^{-1})\right].$$

Since $\varepsilon < A$ we get $(1-\varepsilon)g(J_0) \leqslant g(m) \leqslant (1+\varepsilon)g(J_0)$ and hence $(1-\varepsilon)^2 g(H) < g(m) < (1+\varepsilon)^2 g(H)$, which is what we wanted to prove.

Hence Lim inf $g(n)$ and Lim sup $g(n)$ are finite positive numbers, and their ratio can be forced arbitrarily close to 1 if ε is chosen sufficiently small. Theorem 1(i) follows.

PROOF OF THEOREM 1(ii). Let σ^2 be as in Theorem 1(i). Assume $0 < \gamma < 1$. There are constants $0 < C_1 < C_2$ such that, for all $n = 1, 2, 3, \ldots,$ $C_1 < g(n) < C_2$.

Let N be a positive integer such that $\sum_{n=0}^{\infty} \tau([2^{(N+n)\gamma}]) < A/3$. Let L be a positive integer such that $\sum_{n=0}^{\infty} 2^{1+N+n(\gamma-1)/2} C_2/(C_1 L) < A/3$.

Suppose M is a nonnegative integer and $2^M L^2 \leqslant m \leqslant 2^{M+1}L^2$. For each $n = 0, 1, 2, \ldots$ let $J_n = 2^n m$ and let $K_n = [2^{(N+n)\gamma}]$. Then by Lemma 1,

$$\begin{aligned} g(J_{n+1}) &\geqslant g(J_n)(1-\tau(K_{n+M}+1))^{1/2}\left(1 - 2C_2 K_{n+M}^{1/2}/\left(C_1 J_n^{1/2}\right)\right) \\ &\geqslant g(J_n)(1-\tau(K_{n+M}+1))^{1/2}\left(1 - C_2(C_1L)^{-1}2^{1+N\gamma/2+(M+n)(\gamma-1)/2}\right), \\ g(J_{n+1}) &\leqslant g(J_n)(1+\tau(K_{n+M}+1))^{1/2}\left(1 + 2C_2 K_{n+M}^{1/2}/\left(C_1 J_n^{1/2}\right)\right) \\ &\leqslant g(J_n)(1+\tau(K_{n+M}+1))^{1/2}\left(1 + C_2(C_1L)^{-1}2^{1+N\gamma/2+(M+n)(\gamma-1)/2}\right). \end{aligned}$$

Since $\operatorname{Lim}_{l\to\infty} g(l) = \sigma$ one has

$$g^2(m) \leqslant \sigma^2 \cdot \prod_{n=0}^{\infty}\left[(1-\tau(K_{n+M}+1))^{-1}\left(1 - C_2(C_1L)^{-1}2^{1+N\gamma/2+(M+n)(\gamma-1)/2}\right)^{-2}\right],$$

$$g^2(m) \geqslant \sigma^2 \cdot \prod_{n=0}^{\infty}\left[(1+\tau(K_{n+M}+1))^{-1}\left(1 + C_2(C_1L)^{-1}2^{1+N\gamma/2+(M+n)(\gamma-1)/2}\right)^{-2}\right],$$

and hence by the definition of A,

$$|\sigma^2 - g^2(m)| \leqslant 2\sigma^2\left[\sum_{n=0}^{\infty}\tau(K_{n+M}+1) + 2\sum_{n=0}^{\infty} 2^{1+N\gamma/2+(M+n)(\gamma-1)/2} C_2/(C_1L)\right].$$

If we let $M \to \infty$ and $m \to \infty$ subject to the restriction $2^M L^2 \leqslant m \leqslant 2^{M+1}L^2$, then we get the following:

$$\sum_{n=0}^{\infty} 2^{(M+n)(\gamma-1)/2} = O(2^{M(\gamma-1)/2}) = O(m^{(\gamma-1)/2}),$$

$$\sum_{n=0}^{\infty}\tau(K_{n+M}+1) \leqslant \sum_{n=0}^{\infty}\tau\left(\left[2^{n\gamma}m^{\gamma}/L^2\right]+1\right).$$

Theorem 1(ii) follows.

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