A SUFFICIENT CONDITION FOR LINEAR GROWTH OF VARIANCES IN A STATIONARY RANDOM SEQUENCE

RICHARD C. BRADLEY, JR.

ABSTRACT. Suppose \((X_k, k = \ldots, -1, 0, 1, \ldots)\) is a weakly stationary random sequence. For each positive integer \(n\) let \(S_n = X_1 + \cdots + X_n\) and let \(\tau(n) = \sup|\text{Corr}\left(\sum_{k=-l}^{m} X_k, \sum_{l}^{m+l} X_k\right)|\). If \(\text{Var} S_n \to \infty\) as \(n \to \infty\) and \(\sum_{n=0}^{\infty} \tau(2^n) < \infty\), then \(n^{-1} \text{Var} S_n\) converges to a finite positive limit as \(n \to \infty\). A bound on the rate of convergence is obtained.

Let \((X_k, k = \ldots, -1, 0, 1, \ldots)\) be a weakly stationary sequence of random variables on a probability space \((\Omega, \mathcal{F}, P)\). That is, \(EX_k^2 < \infty\) for all \(k\), there are constants \(\mu\) and \(\gamma(0)\) such that \(EX_k = \mu\) and \(\text{Var} X_k = \gamma(0)\) for all \(k\), and there is a sequence of numbers \(\gamma(1), \gamma(2), \gamma(3), \ldots\) such that \(\text{Cov}(X_k, X_l) = \gamma(|k - l|)\) for all \(k \neq l\). For \(-\infty < J < L < \infty\) let \(H_{J,L}\) denote the \(L^2\)-closure of the linear space spanned by the r.v.'s \(X_k - \mu, J < k < L\). For \(n = 1, 2, 3, \ldots\) let \(S_n = X_1 + X_2 + \cdots + X_n\) and define the quantities

\[
\tau(n) = \sup|\text{Corr}(f, g)|: f \in H_{0,\infty}^0, g \in H_{n,\infty}^0, \quad r(n) = \sup|\text{Corr}(f, g)|: f \in H_{0,\infty}^0, g \in H_{n,\infty}^0,
\]

Obviously the sequences \(\{r(n)\}\) and \(\{\tau(n)\}\) are each nonincreasing, and \(0 < \tau(n) < r(n)\).

Ibragimov and Rozanov [3, Note 2, p. 190] proved the following theorem (it was originally proved in [4]).

**Theorem 0 (Ibragimov and Rozanov).** If \((X_k)\) is weakly stationary and \(\sum_{n=0}^{\infty} r(2^n) < \infty\), then \((X_k)\) has an absolutely continuous spectral distribution function, with a continuous spectral density \(f(\lambda)\).

In [3, Note 2, p. 190] this is stated for stationary Gaussian sequences, but the extension to general weakly stationary sequences is easy. Ibragimov [2, Theorem 2.2] proved a central limit theorem for strictly stationary random sequences satisfying a condition similar to but stronger than \(\sum r(2^n) < \infty\); in that theorem it was assumed that \(f(0) > 0\). In central limit theorems for weakly dependent random variables it is common practice to assume \(\text{Var} S_n \to \infty\) as \(n \to \infty\). Consider the question whether \(f(0) > 0\) follows if one assumes \(\sum r(2^n) < \infty\) and \(\text{Var} S_n \to \infty\).
The answer is affirmative, and one can prove this as a corollary to Theorem 0 by using the results in Chapters 4 and 5 of [3]. It also follows easily from the following theorem.

**Theorem 1.** If \((X_k)\) is weakly stationary, \(\text{Var} \ S_n \to \infty \text{ as } n \to \infty\), and \(\sum_{n=0}^\infty \tau(2^n) < \infty\), then the following statements hold:

(i) There exists a finite positive constant \(\sigma^2\) such that \(\lim_{n \to \infty} \frac{\text{Var} \ S_n}{n} = \sigma^2\).

(ii) For each \(\gamma, 0 < \gamma < 1\), there exist positive constants \(C\) and \(D\) such that for all \(m > 1\),

\[
|\sigma^2 - m^{-1} \text{Var} \ S_m| < C \left( m^{(\gamma - 1)/2} + \sum_{n=0}^\infty \tau\left(\left[\frac{(2^n)^\gamma}{D}\right] + 1\right) \right)
\]

where \([t]\) denotes the greatest integer \(< t\).

Theorem 1 covers the case where \(\tau(n) = o((\log n)^{1+\epsilon})\) for some \(\epsilon > 0\). For the stationary Gaussian sequence given in Example 1 on pp. 179–180 of [3], \(\tau(n) = O((\log n)^{-1})\) and \(n^{-1} \text{Var} \ S_n \to \infty \text{ as } n \to \infty\). Berkes and Philipp [1, Theorem 4] prove an “almost sure invariance principle” for strictly stationary random sequences satisfying the \(\phi\)-mixing condition with a rather slow (logarithmic) mixing rate. Their conditions satisfy the hypothesis of Theorem 1 (see [1, Lemma 4.1.1, p. 44]), and it would be interesting to see if one can choose \(a_n = \sigma^2 n\) in the conclusion of their theorem. In the cases where \(\tau(n) \to 0\) rapidly, other methods generally give much better bounds on \(|\sigma^2 - m^{-1} \text{Var} \ S_m|\) than Theorem 1(ii) does.

**Proof of Theorem 1.** We will assume \(EX_k = 0\) and \(\text{Var} \ X_k = 1\). Let \(S_0 = 0\). For any r.v. \(X\) let \(\|X\| = (EX^2)^{1/2}\). For each \(n = 1, 2, 3, \ldots\) let \(g(n) = n^{-1/2}\|S_n\|\).

**Lemma 1.** If \(N > 1, K > 0, \text{ and } \tau(K + 1) < \frac{1}{2}\), then for \(L = 2N\) and for \(L = 2N + 1\) one has

\[
\|S_N\| \cdot \left[ 2(1 - \tau(K + 1)) \right]^{1/2} \left( 1 - 2b/\|S_N\| \right) < \|S_L\| \leq \|S_N\| \cdot \left[ 2(1 + \tau(K + 1)) \right]^{1/2} \left( 1 + 2b/\|S_N\| \right)
\]

where \(b = \max\{\|S_{K-1}\|, \|S_K\|\}\).

**Proof.**

\[
S_L = S_N + (S_{N+K} - S_N) + (S_{2N+K} - S_{N+K}) - (S_{2N+K} - S_L),
\]

(1)

\[
2(1 - \tau(K + 1))\|S_N\|^2 < \|S_N + S_{2N+K} - S_{N+K}\|^2 < 2(1 + \tau(K + 1))\|S_N\|^2
\]

and Lemma 1 follows from Minkowski's inequality.

**Lemma 2.** Given any \(\epsilon > 0\) and any positive integer \(L_0\), there are positive integers \(N\) and \(L\) with \(L > L_0\) such that \(\forall I, L < i < 2L\), one has

\[
(1 - \epsilon)g(N) < g(i) < (1 + \epsilon)g(N).
\]

**Proof.** For each integer \(J > 2\), \(\text{Corr}(S_N, S_{IN} - S_{(J-I)N}) \to 0\) as \(N \to \infty\). (For \(J = 2\) this just follows from (1).) Hence \(\forall J > 2, g(NJ)/g(N) \to 1\) as \(N \to \infty\).
Let $M$ be a positive integer such that $(1 - \epsilon) < (1 - \epsilon)^{1/2} - M^{-1/2} < (1 + \epsilon)^{1/2} + M^{-1/2} < (1 + \epsilon)$. Let $N$ be such that $MN > L_0$.

$$||S_N|| = \max\{||S_k||, 1 \leq k < N\},$$

and for each $m, M \leq m < 2M$, one has

$$(1 - \epsilon)^{1/2} < g(mN)/g(N) < (1 + \epsilon)^{1/2}.$$ 

Let $L = MN$.

If $L < l < 2L$, then for some $m$ one has $M < m < 2M - 1$ and $mN < l < (m + 1)N$, and hence

$$g(1) < (1 + \epsilon)^{1/2} + M^{-1/2} g(N) < (1 + \epsilon)^{1/2}.$$

and Lemma 2 is proved.

Let $0 < A < 1$ be sufficiently small that if $(\alpha_n, n = 1, 2, \ldots)$ is any sequence of real numbers such that $2\alpha_n < A$ then $|1 - \Pi_n(1 + \alpha_n)| < 2\Sigma_n|\alpha_n|$ and $|1 - \Pi_n(1 + \alpha_n)| < 2\Sigma_n|\alpha_n|$. Let $[\cdot]$ denote the greatest-integer function.

Proof of Theorem 1(i). Assume $0 < \epsilon < A$. Let $N$ be a positive integer such that $\Sigma_{n=0}^{\infty} \tau[2^{n+n/6}] < \epsilon/6$. Let $C > 0$ be such that $\Sigma_{n=0}^{\infty} 2^{n+1-n/6}C^{-1} < \epsilon/6$. Let $L_0$ be a positive integer such that (i) $\forall J > L_0$, $\|S_J\| > C$, (ii) $\Sigma_{n=0}^{\infty} 2^nL_0^{-1} < \epsilon/6$, and (iii) $\forall J > L_0$, $g(2J) > 2^{-1/6}g(J)$ and $g(2J + 1) > 2^{-1/6}g(J)$ (see Lemma 1). For each $n = 0, 1, 2, \ldots$ let $K_n = [2^{n+n/6}]$. Using Lemma 2, let the positive integers $H$ and $L$ be such that $L > L_0$ and $w < L < 2L$, one has $(1 - \epsilon)g(H) < g(I) < (1 + \epsilon)g(H)$.

Let $m$ be an arbitrary positive integer satisfying $m > 2L$. We wish to prove $(1 - \epsilon)^{1/2}g(H) < g(m) < (1 + \epsilon)^{1/2}g(H)$.

For some positive integer $M, 2^ML < m < 2^M + 1L$. There is a sequence of positive integers $J_0, J_1, \ldots, J_M$ such that $m = J_M$, $L < J_0 < 2L$, and for each $n = 0, 1, \ldots, M = 1, J_{n+1} = 2J_n$ or $J_{n+1} = 2J_n + 1$. For each $n = 0, 1, \ldots, M + 1$, one has $||S_j|| > 2^m/3||S_{j0}|| > 2^m/3C$, and using Lemma 1 and the inequality $||S_K|| < K$ one has

$$g(J_{n+1}) > g(J_n)(2J_n/J_n + 1)^{1/2}(1 - \tau(K_n + 1))^{1/2}(1 - 2K_n/(2^n/3C))$$

$$> g(J_n)(2^nL_0^{-1})^{1/2}(1 - \tau(K_n + 1))^{1/2}(1 - 2^{n+1-n/6}C^{-1}),$$

$$g(J_{n+1}) < g(J_n)(1 + \tau(K_n + 1))^{1/2}(1 + 2K_n/(2^n/3C))$$

$$< g(J_n)(1 + \tau(K_n + 1))^{1/2}(1 + 2^{n+1-n/6}C^{-1}).$$

Hence

$$g(m) > g(J_0) \prod_{n=0}^{M-1} [(1 - 2^nL_0^{-1})^{1/2}(1 - \tau(K_n + 1))^{1/2}(1 - 2^{n+1-n/6}C^{-1})],$$

$$g(m) < g(J_0) \prod_{n=0}^{M-1} [(1 + \tau(K_n + 1))^{1/2}(1 + 2^{n+1-n/6}C^{-1})].$$
Since \( \epsilon < A \) we get \( (1 - \epsilon)g(J_0) < g(m) < (1 + \epsilon)g(J_0) \) and hence \( (1 - \epsilon)^2g(H) < g(m) < (1 + \epsilon)^2g(H) \), which is what we wanted to prove.

Hence \( \lim \inf g(n) \) and \( \lim \sup g(n) \) are finite positive numbers, and their ratio can be forced arbitrarily close to 1 if \( \epsilon \) is chosen sufficiently small. Theorem 1(i) follows.

**Proof of Theorem 1(ii).** Let \( \sigma^2 \) be as in Theorem 1(i). Assume \( 0 < \gamma < 1 \). There are constants \( 0 < C_1 < C_2 \) such that, for all \( n = 1, 2, 3, \ldots, C_1 < g(n) < C_2 \).

Let \( N \) be a positive integer such that \( \sum_{n=1}^{\infty} \tau(2^{N+n}) < A/3 \). Let \( L \) be a positive integer such that \( \sum_{n=0}^{\infty} 2^{1+N+n(\gamma-1)/2} C_2/(C_1 L) < A/3 \).

Suppose \( M \) is a nonnegative integer and \( 2^M L^2 < m < 2^{M+1} L^2 \). For each \( n = 0, 1, 2, \ldots \) let \( J_n = 2^m n \) and let \( K_n = [2^{N+n}] \). Then by Lemma 1,

\[
\begin{align*}
g(J_{n+1}) &> g(J_n)(1 - \tau(K_{n+M} + 1))^{1/2}(1 - 2C_2k_2/(C_1j_2^{1/2})) \\
&> g(J_n)(1 - \tau(K_{n+M} + 1))^{1/2}(1 - 2C_2(C_1L)^{-1}2^{1+N\gamma/2+(M+n)(\gamma-1)/2}) \quad (1)
\end{align*}
\]

\[
\begin{align*}
g(J_{n+1}) &< g(J_n)(1 + \tau(K_{n+M} + 1))^{1/2}(1 + 2C_2K_{n+2}/(C_1j_2^{1/2})) \\
&< g(J_n)(1 + \tau(K_{n+M} + 1))^{1/2}(1 + 2C_2(C_1L)^{-1}2^{1+N\gamma/2+(M+n)(\gamma-1)/2}) \quad (2)
\end{align*}
\]

Since \( \lim_{n \to \infty} g(l) = \sigma \) one has

\[
\begin{align*}
g^2(m) &< \sigma^2 \cdot \prod_{n=0}^{\infty} \left[ (1 - \tau(K_{n+M} + 1))^{-1}(1 - 2C_2(C_1L)^{-1}2^{1+N\gamma/2+(M+n)(\gamma-1)/2})^{-2} \right], \\
g^2(m) &> \sigma^2 \cdot \prod_{n=0}^{\infty} \left[ (1 + \tau(K_{n+M} + 1))^{-1}(1 + 2C_2(C_1L)^{-1}2^{1+N\gamma/2+(M+n)(\gamma-1)/2})^{-2} \right]
\end{align*}
\]

and hence by the definition of \( A \),

\[
|\sigma^2 - g^2(m)| < 2\sigma^2 \left[ \sum_{n=0}^{\infty} \tau(K_{n+M} + 1) + 2 \sum_{n=0}^{\infty} 2^{1+N\gamma/2+(M+n)(\gamma-1)/2} C_2/(C_1 L) \right].
\]

If we let \( M \to \infty \) and \( m \to \infty \) subject to the restriction \( 2^M L^2 < m < 2^{M+1} L^2 \), then we get the following:

\[
\begin{align*}
\sum_{n=0}^{\infty} 2^{(M+n)(\gamma-1)/2} &= O(2^{M(\gamma-1)/2}) = O(m^{(\gamma-1)/2}), \\
\sum_{n=0}^{\infty} \tau(K_{n+M} + 1) &\leq \sum_{n=0}^{\infty} \tau(2^N m^{\gamma/L^2} + 1).
\end{align*}
\]

Theorem 1(ii) follows.

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**References**


**Department of Mathematics, Indiana University, Bloomington, Indiana 47405**