AN EXAMPLE IN SHAPE THEORY

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ABSTRACT. We give an example of a compactum which cannot be "improved" within its shape class so that its shape theory agrees with its homotopy theory.

1. Introduction. S. Ferry [2] has proved the following theorem: For every $UV(1)$ compactum $X$ there is an "improved" compactum $X'$ shape equivalent to $X$ such that for every finite-dimensional compactum $Z$ there is a one-to-one correspondence between the set of homotopy classes and the set of strong shape morphisms from $Z$ to $X'$. Two questions arise concerning the hypotheses in that theorem:

(i) Is $UV(1)$ a necessary requirement; i.e. is there a compactum that cannot be improved?

(ii) Is the finite dimensionality of $Z$ essential?

In this paper we give an example to answer in the affirmative the first question.

I have to thank Ross Geoghegan for introducing me to Ferry's work and for his interest and help. In particular, he communicated a version of the folk lemma given in the Appendix. The referee suggested the version he wanted to see published and expressed regret Chapman had not done so in 1970. I am grateful to the referee for changing the original example (so that one would not be misled by the size of $\pi_1$), the Hawaiian earring.

2. The example. Let $X_n$ denote the wedge of a circle with $n$ copies of the $k$-sphere $S^k$, $n > 0$, $k > 2$. Let $X$ be the limit of the sequence $X_0 \leftarrow X_1 \leftarrow \cdots$ with bond $p^n_m: X_n \rightarrow X_m$ equal to the identity on $S^1$ and the first $m$ copies of $S^k$, and sending the remaining $n - m$ copies of $S^k$ to the basepoint. Note that $X_n$ inherits a metric as a subset of $S^1 \vee B^{k+1} \subseteq S^1 \times B^{k+1}$, where $B^{k+1}$ is the $(k + 1)$-dimensional ball.

THEOREM. For any compactum $Y$ shape equivalent to $X$, the natural map

$$[S^k, Y] \rightarrow \{\text{strong shape morphisms from } S^k \text{ to } Y\},$$

is not surjective.

To get an idea of the proof, consider first the case $Y = X$. Let $a_1, \ldots, a_n$ be the obvious generators of $\pi_k(X_n)$ as a module over $\pi_1(X_n) = \mathbb{Z}$ (i.e. $a_i$ is represented by
a map that sends $S^k$ by the identity to the $i$th copy of $S^k$ in $X_n$). We choose a generator $t$ of $\pi_1(X_n)$ and denote the action of $t$ on $\pi_k(X_n)$ by $g \mapsto tg$. Any infinite word of the form

$$a = \sum_{i=1}^{\infty} (t^{k_i} + t^{-k_i})a_i,$$

with $k_i \to \infty$ as $i \to \infty$, determines a shape morphism $S^k \to X$ which cannot be represented by a map. Since there is a forgetful map from strong shape morphisms to shape morphisms, this suffices to prove the theorem.

Let $A$ denote the set of all such $a$'s. It is the size of $A$ that makes the theorem true. For, given any finite subset $A' \subset A$, there is a compactum $X'$ and a shape equivalence $h: X \to X'$ such that for every $a \in A'$, $ha$ is representable by a map: just take $X'$ to be the shape mapping cylinder of $\bigcup_{a \in A'} a: \bigcup_{a \in A'} S^k \to X$.

**Proof of the theorem.** Let $Y$ be a compactum shape equivalent to $X$. It follows from the Appendix that there is a sequence of embeddings $j_n^{n+1}: X_{n+1} \times Q \to X_n \times Q$ ($Q =$ Hilbert cube) such that

$$Y = \lim_{\to} \left( X_0 \times Q \leftarrow X_1 \times Q \leftarrow \cdots \right) = \bigcap_{n=0}^{\infty} j_n^{n}(X_n \times Q)$$

and such that $j_n^{n+1}$ is homotopic to $p_n^{n+1} \times \text{id}_Q$. The natural map from $Y$ to $Y_n = X_n \times Q$ will be denoted by $j_n$.

If $\alpha: S^k \to Y$ is a map, then $j_0\alpha = j_0^n j_0\alpha: S^k \to Y_0 = S^1 \times Q$ is null-homotopic and lifts to the universal cover $R^1 \times Q$ of $S^1 \times Q$. We will construct a shape morphism $\omega: S^k \to Y$ such that if $\omega_n: S^k \to Y_n$ is any map representing the shape morphism $j_0\omega$, then the sequence $\text{diam}\ j_n\omega_n(S^k)$ is unbounded, where $j_0^n\omega_n$ is any lift of $j_0^n\omega_n$ to the universal cover $R^1 \times Q$ of $S^1 \times Q$. This shows that $\omega$ is not representable by a map.

Let $\tilde{Y}_n$ denote the universal cover of $Y_n$. Note that $X_n \times Q = \tilde{X}_n \times Q$ has a metric which agrees locally with the metric on $X_n \times Q$, is invariant under covering translations, and agrees with the usual metric on $R^1 \times B^{k+1} \times Q$. In particular, if $f: A \to X_n$ is a map, then $\text{diam}\ \tilde{f}(A)$ is independent of the choice of the lifting $\tilde{f}: A \to \tilde{X}_n$ of $f$, provided $A$ is path connected.

**Lemma.** If the map $f: X_n \times Q \to X_0 \times Q$ is such that

$$f_n: \pi_1(X_n \times Q) \to \pi_1(X_0 \times Q)$$

is an isomorphism, then for each $N > 0$ there exists an $M > 0$ such that if $A \subset \tilde{X}_n \times Q$ is a subset with $\text{diam}\ A > M$ and $\tilde{f}: \tilde{X}_n \times Q \to \tilde{X}_0 \times Q$ covers $f$, then $\text{diam}\ \tilde{f}(A) > N$.

**Proof.** This follows immediately from the fact that $f$ is proper and from our choice of metrics.

We will now construct the desired shape morphism $\omega: S^k \to Y$. It will be determined by an infinite word of the form $\omega = \Sigma_{i=1}^{\infty} (t^{k_i} + t^{-k_i})a_i$. Each such word defines a shape morphism $\omega: S^k \to Y$ such that $j_n\omega$ is represented by $[\omega_n] = \Sigma_{i=1}^{\infty} (t^{k_i} + t^{-k_i})a_i$. Choose $M_n$ as in the lemma above so that if $\text{diam}\ A > M_n$, then
diam \widehat{\alpha}_n(S^k) > n + 1. Pick \( k_n \) sufficiently large to guarantee that if \( \alpha_n: S^k \rightarrow X_n \times Q \) represents \([\omega_n]\), then \( \alpha_n: S^k \rightarrow \tilde{X}_n \times Q \) has diam \( \tilde{\alpha}_n(S^k) > M_n \). This can be done since diam \( \tilde{\alpha}_n(S^k) > 2k_n \). Thus, diam \( \widehat{\alpha}_n(S^k) > n + 1 \). Suppose there is a map \( \alpha: S^k \rightarrow Y \) representing the constructed shape morphism \( \omega \). We can set \( \alpha_n = j_n \alpha \) and conclude that

\[
\text{diam} \widehat{\alpha}_n(S^k) = \text{diam} \widehat{j_0\alpha_n(S^k)} = \text{diam} \widehat{j_0\alpha_n(S^k)} > n + 1,
\]

for each \( n \). This contradicts the compactness of \( S^k \) and completes the proof.

**Appendix.**

**Lemma.** Let \( A = \lim A_n \) with bonds \( p_n: A_n \rightarrow A_{n-1} \) and with each \( A_n \) a compact ANR. If \( X \) is a compactum shape equivalent to \( A \) then there is a sequence of embeddings \( j_{n+1}: A_{n+1} \times Q \rightarrow A_n \times Q \) such that \( X = \lim (A_n \times Q, j_n) = \bigcap_{n=1}^{\infty} j_i^*(A_n \times Q) \). Moreover, \( j_n \) is homotopic to \( p_n \times \text{id}_Q \).

**Proof.** If \( j: A \times Q \rightarrow Q \) is a Z-embedding, then \( j(A \times Q) = \bigcap_{i=1}^{\infty} M_i \), where each \( M_i \) is a \( Q \)-manifold neighborhood of \( A \times Q \) homeomorphic to \( A \times Q \) in such a way that the diagrams

\[
\begin{array}{ccc}
M_{i+1} & \rightarrow & M_i \\
\downarrow & & \downarrow \\
A_{i+1} \times Q & \xrightarrow{p_{i+1} \times \text{id}_Q} & A_i \times Q
\end{array}
\]

commute up to homotopy. This is well known. See [1], for example, for a proof.

Since \( A \times Q \) and \( X \) are shape equivalent, the proof of Chapman’s complement theorem [1] produces an isotopy \( f_t: Q \rightarrow Q \), \( 0 < t < 1 \), such that \( f_t \) and \( f_t^{-1} \) are supported on smaller and smaller neighborhoods of \( A \times Q \) and \( X \), respectively. If \( \{t_i\} \) is a sequence of real numbers, \( 0 < t_i < 1 \), converging rapidly to \( 1 \), \( X = \bigcap_{i=1}^{\infty} f_{t_i}(M_i) \) and \( f_{t_i}|M_{i+1} \) is ambient isotopic to \( f_{t_{i+1}}|M_{i+1} \) in \( M_i \). This shows not only that neighborhoods of \( A \) are homeomorphic (simple homotopy equivalent) to neighborhoods of \( X \) but also that the homeomorphisms can be chosen coherently.

**References**


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