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A RIGID SPACE X FOR WHICH $X \times X$ IS HOMOGENEOUS; AN APPLICATION OF INFINITE-DIMENSIONAL TOPOLOGY

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ABSTRACT. We give an example of a rigid (= no autohomeomorphisms beyond the identity) space X such that $X \times X$ is homogeneous. In fact, $X \times X$ is homeomorphic to the Hilbert cube. This answers a question of A. V. Arhangel'skiĭ.

1. Introduction. In his survey paper [**1**, p. 59], A. V. Arhangel'skiĭ asks whether there is an example of a nonhomogeneous (compact) space whose square is homogeneous. The aim of this note is to show that recent results in infinite-dimensional topology can be used to construct a rigid (= no autohomeomorphisms beyond the identity) space X such that $X \times X \approx Q$, the Hilbert cube. As a consequence, $X \times X$ is homogeneous. Modulo some very nontrivial known facts it is not difficult to construct X. Since X will prevent general topologists from asking certain natural questions concerning homogeneity, we feel that our example should appear in print.

For all undefined terms concerning infinite-dimensional topology, see [**2**]. I am indebted to Jeroen Bruyning and to Marcel van de Vel for some helpful comments.

2. Embedding $M \times Q \times [0, 1)$ in Q. Wong [**12**] has shown that wild Cantor sets in Q exist. In [**4**] Daverman obtained, among others, the same result by totally different methods. We will apply Daverman's construction to obtain an embedding theorem for certain Q-manifolds in Q.

2.1. THEOREM. *Let M be a compact acyclic, PL n-manifold (with boundary). Then there is a finite-dimensional compact AR $B \subset Q$ such that $Q - B \approx M \times Q \times [0, 1)$.*

PROOF. Let $X \subset M$ be an $(n-1)$-spine for M and fix a Cantor set C in the interior of the 2-cell I^2 and take $q \in Q$ arbitrarily. Daverman [**4**] shows that $Q^* = (M \times I^2 \times Q)/G$, where G is the upper semicontinuous decomposition of $M \times I^2 \times Q$ having $\{X \times \{c\} \times \{q\}: c \in C\}$ for its set of nondegenerate elements, is homeomorphic to Q. Let $\pi: M \times I^2 \times Q \to Q^*$ be the decomposition map. Put $B = \pi(M \times I^2 \times \{q\})$. By [**3**], B is an AR which, by the countable sum

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theorem, is finite-dimensional. Clearly, $Q^* - B = \pi(M \times I^2 \times (Q - \{q\})) \approx M \times I^2 \times (Q - \{q\}) \approx M \times I^2 \times Q \times [0, 1) \approx M \times Q \times [0, 1)$. (This proof can be simplified of course. Daverman's construction proves more than we need.) □

We will now show that the AR of Theorem 2.1 can be chosen to be of arbitrarily small diameter.

2.2. LEMMA. *Let $A \subset Q$ be closed and nowhere dense, and let $Z \subset Q$ be a Z-set. Then there is a homeomorphism $h: Q \to Q$ such that $Z \cap h(A) = \varnothing$.*

PROOF. Since $A \subset Q$ is nowhere dense, there is a Z-set $Z' \subset Q - (A \cup Z)$ which is homeomorphic to Z. Let $f: Z' \to Z$ be an arbitrary homeomorphism. By the Homeomorphism Extension Theorem [2], there exists a homeomorphism $h: Q \to Q$ extending f. It is clear that h is as required. □

2.3. COROLLARY. *Let M be a compact acyclic, PL n-manifold (with boundary) and let $U \subset Q$ be open and nonempty. Then there is a finite-dimensional compact AR $B \subset U$ such that $Q - B \approx M \times Q \times [0, 1)$.*

PROOF. Find an $n \geqslant 1$ and nondegenerate intervals $[a_i, b_i] \subset I$ $(i \leqslant n)$ such that

$$Q' = [a_1, b_1] \times \cdots \times [a_n, b_n] \times I \times I \times \cdots \subset U.$$

Let $S = \{x \in Q' \colon \exists\, i \leqslant n \text{ with } x_i \in \{a_i, b_i\}\}$. It is clear that S is a Z-set in the Hilbert cube Q'. By Theorem 2.1 and Lemma 2.2 we can find a finite-dimensional compact AR $B \subset Q' - S$ such that $Q' - B \approx M \times Q \times [0, 1)$. Clearly, $Q - B \approx Q' - B$. □

3. The example. In this section, M denotes a fixed compact, acyclic, nonsimply connected PL n-manifold (with boundary). Corollary 2.3 implies that we can find a family $\{K_i \colon i \in \mathbf{N}\}$ of finite-dimensional compact AR's in Q such that

(1) $i < j \to K_i \cap K_j = \varnothing$,

(2) $\operatorname{diam}(K_i) < 2^{-i}$,

(3) $\bigcup_1^\infty K_i$ is dense in Q, and

(4) $Q - K_i \approx M^i \times Q \times [0, 1)$.

Let G be the upper semicontinuous decomposition of Q having $\{K_i \colon i \in \mathbf{N}\}$ for its set of nondegenerate elements and put $X = Q/G$. Let $\pi: Q \to X$ be the decomposition. We claim that X is as required.

3.1. LEMMA. *X is an AR.*

PROOF. Since π is cell-like and has only countably many nondegenerate point-inverses, this follows from Kozlowski [6]. □

3.2. LEMMA. *X is rigid.*

PROOF. Since π is cell-like and since X is an AR, for each $x \in X$ it is true that $X - \{x\}$ and $Q - \pi^{-1}(x)$ have the same homotopy type, Haver [5]. Let x_i be the unique point in $\pi(K_i)$ $(i \in \mathbf{N})$. Since M is nonsimply connected and since the fundamental group $\pi_1(M)$ is finitely generated [**9**, p. 141], it follows that $\pi_1(M^i) \not\approx \pi_1(M^j)$ if $i \neq j$. Simply observe that

$$\pi_1(M^i) \approx \underbrace{\pi_1(M) * \pi_1(M) * \cdots * \pi_1(M)}_{i \text{ times}}$$

and that, by Gruško's Theorem [**8**, p. 58], this implies that the minimum number of generators of $\pi_1(M^i)$ is equal to i times the minimum number of generators of $\pi_1(M)$. Since $\pi_1(M^i) \approx \pi_1(Q - K_i) \approx \pi_1(X - \{x_i\})$ for all $i \in \mathbf{N}$, we can therefore conclude that no autohomeomorphism of X can map some x_i onto x_j for $i \neq j$. Now take $x \in X - \{x_i: i \in \mathbf{N}\}$. Then $\pi_1(X - \{x\}) \approx \pi_1(Q - \{\mathrm{pt}\}) = 0$. Therefore, for the same reason, no autohomeomorphism of X can map x_i onto a point of $X - \{x_i: i \in \mathbf{N}\}$. We conclude that each autohomeomorphism of X is the identity on the dense set $\{x_i: i \in \mathbf{N}\}$, hence must be the identity. $\square$

3.3. Lemma. $X \times X \approx Q$.

Proof. Since X is an AR by Lemma 3.1, and since π is a cell-like map having only countably many nondegenerate, finite-dimensional point-inverses, this follows from Toruńczyk [**10**, Theorem 5]. $\square$

4. Remarks. Since our example is infinite-dimensional, one naturally wonders whether a finite-dimensional space exists with similar properties. I do not know the answer to this question. In addition, we have seen, as is well known, that one can use connectivity properties to construct rigid spaces. This type of construction does not work in the zero-dimensional case. In fact, it is much more complicated to construct a rigid zero-dimensional space than a rigid continuum. For details, see Kuratowski [7]. Using results of [9] we can construct a homogeneous subset of the real line which is the union of countably many dense rigid subspaces, but we do not know whether this space admits a product structure. I think it would be very interesting to have a rigid zero-dimensional space with a homogeneous square. One is also naturally led to the question whether there is a rigid space X for which $X \times X$ is a topological group. In conclusion let us pose the following problems (all spaces are separable metric):

(1) *Is there a finite-dimensional rigid continuum X for which $X \times X$ is homogeneous?*

(2) *Is there a rigid space X such that $X \times X$ is a topological group? Can X be a continuum? zero-dimensional?*

(3) *Is there a rigid zero-dimensional space X for which $X \times X$ is homogeneous?*

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