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A NOTE ON WALLMAN COMPACTIFICATIONS

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ABSTRACT. T_3 and $T_{3\frac{1}{2}}$ spaces are characterized in terms of a Wallman compactification of a T_1 space X.

For a T_1 space $(X, \mathfrak{T})$ consider the Wallman compactification $(\chi, (X^*, \mathfrak{W}))$ [3] consisting of the set X^* of all ultraclosed filters on $(X, \mathfrak{T})$, the topology $\mathfrak{W}$ on X^* generated by $\{U^*: U \in \mathfrak{T}\}$ where $U^* = \{\mathfrak{F} \in X^*: U \in \mathfrak{F}\}$, and the dense embedding $\chi: X \to X^*$ defined by setting $\chi(x) = \mathfrak{S}(x) = \{A \subset X: x \in A\}$. A well-known result about $(X^*, \mathfrak{W})$ is that $(X, \mathfrak{T})$ is T_4 iff $(X^*, \mathfrak{W})$ is T_2. Here we characterize T_3 and $T_{3\frac{1}{2}}$ spaces in a similar manner.

We define a space $(Y, \mathfrak{U})$ to be T_2 *relative to* X for a subset X of Y if for each $x \in X$ and for each $y \in Y$ with $x \neq y$ there exist disjoint $\mathfrak{U}$-open sets U and V such that $x \in U$ and $y \in V$. $(Y, \mathfrak{U})$ is called *completely* T_2 *relative to* X if for $x \in X$ and $y \in Y$ with $x \neq y$, there exists a continuous real-valued function for Y with $f(x) \neq f(y)$.

THEOREM 1. *X is $T_{3\frac{1}{2}}$ iff X^* is completely T_2 relative to $\chi(X)$.*

PROOF. If X^* is completely T_2 relative to $\chi(X)$, then X is completely T_2. Let F be a closed subset of X and let $x \notin F$. Since χ is an embedding, $\chi(x) \notin \mathfrak{W}\text{-cl}\, \chi(F)$. As X^* is completely T_2 relative to $\chi(X)$, for each $y \in \mathfrak{W}\text{-cl}\, \chi(F)$, there exist disjoint cozero sets U_y and V_y in X^* such that $y \in U_y$, $\chi(x) \in V_y$. Further $\mathfrak{W}\text{-cl}\, \chi(F)$, being a closed subset of X^*, is compact. Let $\{U_{y_1}, U_{y_2}, \ldots, U_{y_n}\}$ be a finite subcover of $\{U_y: y \in \mathfrak{W}\text{-cl}\, \chi(F)\}$. Then $\bigcup_{i=1}^n U_{y_i}$ and $\bigcap_{i=1}^n V_{y_i}$ are disjoint cozero sets containing $\chi(F)$ and $\chi(x)$. Thus x and F are contained in disjoint cozero subsets of X and, hence, X is completely regular.

Conversely, let X be $T_{3\frac{1}{2}}$ and let $\mathfrak{F} \in X^*$ and $\chi(x) \in \chi(X)$ be such that $\chi(x) \neq \mathfrak{F}$. Since $\mathfrak{F}$ is a closed ultrafilter, we have a closed subset $F \in \mathfrak{F}$ such that $x \notin F$. Let a continuous $g: X \to [0, 1]$ separate x and F. If $g^*: X^* \to [0, 1]$ is the continuous extension of g, then g^* separates $\mathfrak{F}$ and $\chi(x)$.

Using similar arguments one can prove

THEOREM 2. *X is T_3 iff X^* is T_2 relative to $\chi(X)$.*

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In terms of the new definitions here the above result about T_4 spaces can be put down as

THEOREM 3. *X is T_4 iff X^* is T_2 relative to X^*.*

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