A NOTE ON WALLMAN COMPACTIFICATIONS

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Abstract. T3 and T3½ spaces are characterized in terms of a Wallman compactification of a T1 space X.

For a T1 space \((X, \mathcal{T})\) consider the Wallman compactification \((X, (X^*, \mathcal{U}))\) [3] consisting of the set \(X^*\) of all ultraclosed filters on \((X, \mathcal{T})\), the topology \(\mathcal{U}\) on \(X^*\) generated by \(\{U^*: U \in \mathcal{T}\}\) where \(U^* = \{F \in X^*: U \in F\}\), and the dense embedding \(\chi: X \rightarrow X^*\) defined by setting \(\chi(x) = \mathcal{S}(x) = \{A \subset X: x \in A\}\). A well-known result about \((X^*, \mathcal{U})\) is that \((X, \mathcal{T})\) is T4 iff \((X^*, \mathcal{U})\) is T2. Here we characterize T3 and T3½ spaces in a similar manner.

We define a space \((Y, \mathcal{Q})\) to be T2 relative to \(X\) for a subset \(X\) of \(Y\) if for each \(x \in X\) and for each \(y \in Y\) with \(x \neq y\) there exist disjoint \(\mathcal{Q}\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\). \((Y, \mathcal{Q})\) is called completely T2 relative to \(X\) if for \(x \in X\) and \(y \in Y\) with \(x \neq y\), there exists a continuous real-valued function for \(Y\) with \(f(x) \neq f(y)\).

Theorem 1. \(X\) is T3½ iff \(X^*\) is completely T2 relative to \(\chi(x)\).

Proof. If \(X^*\) is completely T2 relative to \(\chi(x)\), then \(X\) is completely T2. Let \(F\) be a closed subset of \(X\) and let \(x \notin F\). Since \(\chi\) is an embedding, \(\chi(x) \notin \mathcal{U}\)-cl \(\chi(F)\). As \(X^*\) is completely T2 relative to \(\chi(x)\), for each \(y \in \mathcal{U}\)-cl \(\chi(F)\), there exist disjoint cozero sets \(U_y\) and \(V_y\) in \(X^*\) such that \(y \in U_y\) and \(\chi(x) \in V_y\). Further \(\mathcal{U}\)-cl \(\chi(F)\), being a closed subset of \(X^*\), is compact. Let \(\{U_{y_1}, U_{y_2}, \ldots, U_{y_n}\}\) be a finite subcover of \(\{U_y: y \in \mathcal{U}\)-cl \(\chi(F)\}\). Then \(\bigcup_{i=1}^{n} U_{y_i}\) and \(\bigcap_{i=1}^{n} V_{y_i}\) are disjoint cozero sets containing \(\chi(F)\) and \(\chi(x)\). Thus \(x\) and \(F\) are contained in disjoint cozero subsets of \(X\) and, hence, \(X\) is completely regular.

Conversely, let \(X\) be T3½ and let \(\mathcal{F} \in X^*\) and \(\chi(x) \in \chi(X)\) such that \(\chi(x) \notin \mathcal{F}\). Since \(\mathcal{F}\) is a closed ultrafilter, we have a closed subset \(F \in \mathcal{F}\) such that \(x \notin F\). Let a continuous \(g: X \rightarrow [0, 1]\) separate \(x\) and \(F\). If \(g^*: X^* \rightarrow [0, 1]\) is the continuous extension of \(g\), then \(g^*\) separates \(\mathcal{F}\) and \(\chi(x)\).

Using similar arguments one can prove

Theorem 2. \(X\) is T3 iff \(X^*\) is T2 relative to \(\chi(x)\).
In terms of the new definitions here the above result about $T_4$ spaces can be put down as

**Theorem 3.** $X$ is $T_4$ iff $X^*$ is $T_2$ relative to $X^*$.

**References**


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