A NOTE ON WALLMAN COMPACTIFICATIONS

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Abstract. T₃ and T₃½ spaces are characterized in terms of a Wallman compactification of a T₁ space X.

For a T₁ space (X, T) consider the Wallman compactification (X*, (X*, W)) consisting of the set X* of all ultraclosed filters on (X, T), the topology W on X* generated by \{U*: U ∈ T\} where U* = (F ∈ X*: U ∈ F), and the dense embedding \(\chi: X \rightarrow X^*\) defined by setting \(\chi(x) = \mathcal{S}(x) = \{A \subseteq X: x \in A\}\). A well-known result about (X*, W) is that (X, T) is T₄ iff (X*, W) is T₂. Here we characterize T₃ and T₃½ spaces in a similar manner.

We define a space (Y, Q) to be T₂ relative to X for a subset X of Y if for each \(x \in X\) and for each \(y \in Y\) with \(x \neq y\) there exist disjoint Q-open sets U and V such that \(x \in U\) and \(y \in V\). (Y, Q) is called completely T₂ relative to X if for \(x \in X\) and \(y \in Y\) with \(x \neq y\), there exists a continuous real-valued function for Y with \(f(x) \neq f(y)\).

**Theorem 1.** X is T₃½ iff X* is completely T₂ relative to \(\chi(X)\).

**Proof.** If X* is completely T₂ relative to \(\chi(X)\), then X is completely T₂. Let F be a closed subset of X and let \(x \notin F\). Since \(\chi\) is an embedding, \(\chi(x) \notin Q_{\text{cl}} \chi(F)\). As X* is completely T₂ relative to \(\chi(X)\), for each \(y \in Q_{\text{cl}} \chi(F)\), there exist disjoint cozero sets \(U_y\) and \(V_y\) in X* such that \(y \in U_y\) and \(\chi(x) \in V_y\). Further \(Q_{\text{cl}} \chi(F)\), being a closed subset of X*, is compact. Let \(\{U_{y_1}, U_{y_2}, \ldots, U_{y_r}\}\) be a finite subcover of \(\{y \in Q_{\text{cl}} \chi(F)\}\). Then \(\bigcup_{y=1}^{r} U_{y}\) and \(\bigcap_{y=1}^{r} V_{y}\) are disjoint cozero sets containing \(\chi(F)\) and \(\chi(x)\). Thus x and \(F\) are contained in disjoint cozero subsets of X and, hence, X is completely regular.

Conversely, let X be T₃½ and let \(\mathcal{F} \in X^*\) and \(\chi(x) \in \chi(X)\) be such that \(\chi(x) \neq \mathcal{F}\). Since \(\mathcal{F}\) is a closed ultrafilter, we have a closed subset \(F \subseteq \mathcal{F}\) such that \(x \notin F\). Let a continuous \(g: X \rightarrow [0, 1]\) separate \(x\) and \(F\). If \(g*: X^* \rightarrow [0, 1]\) is the continuous extension of \(g\), then \(g*\) separates \(\mathcal{F}\) and \(\chi(x)\).

Using similar arguments one can prove

**Theorem 2.** X is T₃ iff X* is T₂ relative to \(\chi(X)\).
In terms of the new definitions here the above result about $T_4$ spaces can be put down as

**Theorem 3.** $X$ is $T_4$ iff $X^*$ is $T_2$ relative to $X^*$.

**References**


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