DIFFERENTIAL EQUATIONS WHICH ARE
TOPOLOGICALLY LINEAR

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ABSTRACT. We show that if the flow \((R, X, \tau)\) defined by an autonomous system
\(\dot{x} = f(x)\) on a closed region \(X\) of \(R^m\) satisfies (i) it is positively nonexpansive, (ii) \(X\)
contains a globally asymptotically stable compact invariant subset which is a
manifold, then there exists an integer \(n\) so that the flow \((R, X, \tau)\) can be
topologically and equivariantly embedded into the flow generated by a linear
system \(\dot{y} = Ay\) where \(A\) is a constant \(n \times n\) matrix.

1. Introduction. By a flow \((R, X, \tau)\) on a topological space \(X\) we mean a
continuous group action \(\tau: R \times X \to X\) of the additive group of reals \(R\) on the
space \(X\). A stable compact subset \(M\) of \(X\) is called globally asymptotically stable if
for any neighborhood \(U\) of \(M\) and for any \(x \in X\) there exists a \(T \in R\) such that
\(\tau([T, \infty), x) \subset U\).

We say that a flow \((R, X_1, \tau_1)\) can be embedded into another flow \((R, X_2, \tau_2)\) if
there exists a topological embedding \(i: X_1 \to X_2\) which is equivariant, i.e., such that
it makes the diagram

\[
\begin{array}{ccc}
R \times X_1 & \xrightarrow{\pi_1} & X_1 \\
I \times \downarrow i & & \downarrow i \\
R \times X_2 & \xrightarrow{\pi_2} & X_2
\end{array}
\]

commutative where \(I\) is the identity mapping on \(R\). If the space \(X\) carries a linear
structure we say that the flow \((R, X, \tau)\) is linear if all the transition functions
\(\tau(t, \cdot): X \to X\) are linear operators. Given an arbitrary constant \(c \in (0, 1)\) we
define the radial flow \((R, R^m, \rho)\) on the Euclidean space \(R^m\) by \(\rho(t, x) = c^t x\) for
t \(\in R\) and \(x \in R^m\). If the flow \((R, X, \tau)\) is defined on a metric space \((X, d)\) we say
that it is positively nonexpansive if for every \(x, y \in X\) and \(t > 0\) we have
d([\tau(t, x), \tau(t, y)] \leq d(x, y). The purpose of this note is to establish the following
statement.

THEOREM. Let a flow \((R, X, \tau)\) be defined on a closed region \(X\) of the Euclidean
space \(R^m\) satisfying the conditions: (i) The flow is positively nonexpansive relative to
the Euclidean metric on \(X\). (ii) The set \(X\) contains a compact manifold \(M\) which is
invariant and globally asymptotically stable in the flow \((R, X, \tau)\).

Then there exists an integer \(n > 1\) such that the flow \((R, X, \tau)\) can be embedded
into a linear flow on \(R^n\).
2. Auxiliary results.

**Theorem A (R. C. McCann).** Let \((R, X, \pi)\) be a flow on an \(n\)-dimensional locally compact metric space containing a globally asymptotically stable critical point. Then \((R, X, \pi)\) can be embedded into the radial flow \((R, R^{2n}, \rho)\).

For the proof see [4].

**Theorem B (R. Sine).** Let a flow \((R, X, \pi)\) on a separable metric space \((X, d)\) satisfy the conditions (i) and (ii) of our theorem (except that \(M\) need not be a manifold). Then there exists a retraction \(a: X \to M\) commuting with \(\pi(t, \cdot)\) for every \(t \in R\).

**Proof.** From (ii) it follows that for every \(x \in X\) the positive semi-orbit \((\pi(t, x): t > 0)\) is precompact. This shows that the hypotheses of Theorem 4 of [7] are met in our case. (See also [1] and [2] for the existence of retractions in the case of a discrete semiflow.) The value \(a(x) \in M\) for \(x \in X\) is the uniquely determined asymptotic point to \(x\), i.e., such that \(d(\pi(t, x), \pi(t, a(x))) \to 0\) as \(t \to +\infty\). (The proof of uniqueness of \(a(x)\) is also given in [7].)

**Theorem C (S. B. Myers, N. Steenrod).** The group \(I(M)\) of isometries of a Riemannian manifold \(M\) is a Lie transformation group with respect to the compact-open topology. If \(M\) is compact, \(I(M)\) is also compact.

For the proof see [3, Theorem 1.2, p. 39].

**Theorem D (G. D. Mostow).** The operation of any compact Lie group of homeomorphisms on a compact manifold can be represented as the restriction to a submanifold of Euclidean space of the operation of some subgroup of the orthogonal group.

For the proof see [5].

3. Proof of the theorem. From the given flow \((R, X, \pi)\) we obtain in a natural way two other flows: \((R, X/M, \pi^*)\) and \((R, M, \pi_1)\) where \(X/M\) is the quotient space obtained from \(X\) by identifying \(M\) to a point and endowed with the quotient topology, \(\pi^*\) is the uniquely defined action which makes the diagram

\[
\begin{array}{ccc}
R \times X & \xrightarrow{\pi} & X \\
I \times a \downarrow & & \downarrow a \\
R \times X/M & \xrightarrow{\pi^*} & X/M
\end{array}
\]

commutative where \(a: X \to X/M\) is the natural projection and \(I\) the identity on \(R\); the action \(\pi_1\) is the restriction of \(\pi\) to \(R \times M\). Since \(X\) is locally compact and \(M\) compact it follows readily that \(X/M\) is again a locally compact metric space of dimension \(m\) (the same as \(X\)).
Lemma 3.1. The flow \((R, X, \pi)\) can be embedded into the cartesian product \((R, X/M, \pi^*) \times (R, M, \pi_1)\).

Proof. Applying Theorem B we construct the embedding \(i: X \to X/M \times M\) setting \(i(x) = (ax, a(x))\) where \(a\) is the natural projection and \(a\) is the retraction guaranteed by Theorem B. The fact that \(i\) is equivariant follows from commutativity between \(a\) and \(\pi(t, \cdot)\).

We now show that both factors, namely \((R, X/M, \pi^*)\) and \((R, M, \pi_1)\) can be embedded in a linear flow on Euclidean spaces. Lemma 3.1 then will imply that the original flow \((R, X, \pi)\) can be embedded in some linear flow on a Euclidean space.

Since \(X/M\) is a locally compact \(m\)-dimensional metric space and the flow \((R, X/M, \pi^*)\) has a globally asymptotically stable critical point, namely \(a(M)\), Theorem A assures us that \((R, X/M, \pi^*)\) can be embedded in a radial flow on \(R^{2m}\).

From the nonexpansiveness of \(\pi(t, \cdot)\) and the compactness of \(M\) it follows that the family \(\{\tau_\tau(t, \cdot): t \in R\}\) of transition functions consists of isometries. Since \(M\) is a manifold located in \(R^m\) it inherits the Riemannian structure from the Euclidean structure of \(R^m\) and Theorem C is therefore applicable. This means that the family \(\{\tau_\tau(t, \cdot): t \in R\}\) is a subgroup of the compact Lie group \(I(M)\) (the group of all isometries of \(M\) endowed with the topology of uniform convergence). But now Theorem D applies to the action of \(I(M)\) on \(M\) implying that \((R, M, \pi_1)\) can be embedded in a linear flow (where the transition functions are in fact orthogonal transformations) on some Euclidean space \(R^n\). This accomplishes the proof of our theorem, since every linear flow on \(R^n\) can be obtained as the flow generated by the equation \(\dot{x} = Ax\) where \(A\) is a constant matrix.

As a simple example let \(X\) be the subset of the plane \(R^2\) with the open disc removed, i.e., in polar coordinates \(X = \{(r, \phi): r > 1\}\) and let \((R, X, \pi)\) be the flow on \(X\) defined by the autonomous system

\[
\frac{dr}{dt} = f(r), \quad \frac{d\phi}{dt} = 1
\]

where \(f\) is a function continuous and decreasing on \([1, \infty)\), negative on \((1, \infty)\) and satisfying \(f(1) = 0\).

The trajectories of this flow spiral around the unit circle \(C\) and approach it as \(t \to \infty\); thus \(C\) is evidently the manifold which is globally and asymptotically stable. The nonexpansiveness of \(\pi\) follows from negativeness of \(f\). Thus, the flow \((R, X, \pi)\) satisfies the conditions of our theorem. We observe that \(X/C\) is homeomorphic to \(R^2\) so that the flow \((R, X/C, \pi^*)\) can be linearized in \(R^4\) and the flow \((R, C, \pi_1)\) which is a rotation can of course be represented in \(R^2\). (In [4] it is shown, see Corollary 9 of this paper, that in this particular case the flow \((R, X/C, \pi^*)\) can be linearized also in \(R^2\).)

Remark. The treatment of the linearization problem as we offer it in this note is of highly nonconstructive character. Also, we are so far unable to produce any a priori estimate of the dimension \(n\) of the linearization space \(R^n\) from the knowledge of the flow \((R, X, \pi)\) and the dimension of its underlying space \(X\).
REFERENCES