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DIFFERENTIAL EQUATIONS WHICH ARE TOPOLOGICALLY LINEAR

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ABSTRACT. We show that if the flow (R, X, π) defined by an autonomous system $\dot{x} = f(x)$ on a closed region X of R^m satisfies (i) it is positively nonexpansive, (ii) X contains a globally asymptotically stable compact invariant subset which is a manifold, then there exists an integer n so that the flow (R, X, π) can be topologically and equivariantly embedded into the flow generated by a linear system $\dot{y} = Ay$ where A is a constant $n \times n$ matrix.

1. Introduction. By a flow (R, X, π) on a topological space X we mean a continuous group action π: $R \times X \to X$ of the additive group of reals R on the space X. A stable compact subset M of X is called globally asymptotically stable if for any neighborhood U of M and for any $x \in X$ there exists a $T \in R$ such that $\pi([T, \infty), x) \subset U$.

We say that a flow (R, X_1, π_1) can be embedded into another flow (R, X_2, π_2) if there exists a topological embedding i: $X_1 \to X_2$ which is equivariant, i.e., such that it makes the diagram

$$\begin{array}{ccc} R \times X_1 & \xrightarrow{\pi_1} & X_1 \\ I \times i \downarrow & & \downarrow i \\ R \times X_2 & \xrightarrow{\pi_2} & X_2 \end{array}$$

commutative where I is the identity mapping on R. If the space X carries a linear structure we say that the flow (R, X, π) is linear if all the transition functions $\pi(t, \cdot)$: $X \to X$ are linear operators. Given an arbitrary constant $c \in (0, 1)$ we define the radial flow (R, R^m, ρ) on the Euclidean space R^m by $\rho(t, x) = c^t x$ for $t \in R$ and $x \in R^m$. If the flow (R, X, π) is defined on a metric space (X, d) we say that it is positively nonexpansive if for every $x, y \in X$ and $t \geqslant 0$ we have $d[\pi(t, x), \pi(t, y)] \leqslant d(x, y)$. The purpose of this note is to establish the following statement.

THEOREM. *Let a flow (R, X, π) be defined on a closed region X of the Euclidean space R^m satisfying the conditions*: (i) *The flow is positively nonexpansive relative to the Euclidean metric on X.* (ii) *The set X contains a compact manifold M which is invariant and globally asymptotically stable in the flow (R, X, π).*

Then there exists an integer $n \geqslant 1$ such that the flow (R, X, π) can be embedded into a linear flow on R^n.

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2. Auxiliary results.

THEOREM A (R. C. MCCANN). *Let (R, X, π) be a flow on an n-dimensional locally compact metric space containing a globally asymptotically stable critical point. Then (R, X, π) can be embedded into the radial flow (R, R^{2n}, ρ).*

For the proof see [4].

THEOREM B (R. SINE). *Let a flow (R, X, π) on a separable metric space (X, d) satisfy the conditions* (i) *and* (ii) *of our theorem (except that M need not be a manifold). Then there exists a retraction a: $X \to M$ commuting with $\pi(t, \cdot)$ for every $t \in R$.*

PROOF. From (ii) it follows that for every $x \in X$ the positive semiorbit $\{\pi(t, x)$: $t \geqslant 0\}$ is precompact. This shows that the hypotheses of Theorem 4 of [7] are met in our case. (See also [1] and [2] for the existence of retractions in the case of a discrete semiflow.) The value $a(x) \in M$ for $x \in X$ is the uniquely determined asymptotic point to x, i.e., such that $d(\pi(t, x), \pi(t, a(x))) \to 0$ as $t \to +\infty$. (The proof of uniqueness of $a(x)$ is also given in [7].)

THEOREM C (S. B. MYERS, N. STEENROD). *The group $I(M)$ of isometries of a Riemannian manifold M is a Lie transformation group with respect to the compact-open topology. If M is compact, $I(M)$ is also compact.*

For the proof see [3, Theorem 1.2, p. 39].

THEOREM D (G. D. MOSTOW). *The operation of any compact Lie group of homeomorphisms on a compact manifold can be represented as the restriction to a submanifold of Euclidean space of the operation of some subgroup of the orthogonal group.*

For the proof see [5].

3. Proof of the theorem. From the given flow (R, X, π) we obtain in a natural way two other flows: $(R, X/M, \pi^*)$ and (R, M, π_1) where X/M is the quotient space obtained from X by identifying M to a point and endowed with the quotient topology, π^* is the uniquely defined action which makes the diagram

$$\begin{array}{ccc} R \times X & \xrightarrow{\pi} & X \\ I \times \alpha\downarrow & & \downarrow\alpha \\ R \times X/M & \xrightarrow{\pi^*} & X/M \end{array}$$

commutative where α: $X \to X/M$ is the natural projection and I the identity on R; the action π_1 is the restriction of π to $R \times M$. Since X is locally compact and M compact it follows readily that X/M is again a locally compact metric space of dimension m (the same as X).

LEMMA 3.1. *The flow* (R, X, π) *can be embedded into the cartesian product* $(R, X/M, \pi^*) \times (R, M, \pi_1)$.

PROOF. Applying Theorem B we construct the embedding i: $X \to X/M \times M$ setting $i(x) = (\alpha x, a(x))$ where α is the natural projection and a is the retraction guaranteed by Theorem B. The fact that i is equivariant follows from commutativity between a and $\pi(t, \cdot)$.

We now show that both factors, namely $(R, X/M, \pi^*)$ and (R, M, π_1) can be embedded in a linear flow on Euclidean spaces. Lemma 3.1 then will imply that the original flow (R, X, π) can be embedded in some linear flow on a Euclidean space.

Since X/M is a locally compact m-dimensional metric space and the flow $(R, X/M, \pi^*)$ has a globally asymptotically stable critical point, namely $\alpha(M)$, Theorem A assures us that $(R, X/M, \pi^*)$ can be embedded in a radial flow on R^{2m}.

From the nonexpansiveness of $\pi(t, \cdot)$ and the compactness of M it follows that the family $\{\pi_1(t, \cdot): t \in R\}$ of transition functions consists of isometries. Since M is a manifold located in R^m it inherits the Riemannian structure from the Euclidean structure of R^m and Theorem C is therefore applicable. This means that the family $\{\pi_1(t, \cdot): t \in R\}$ is a subgroup of the compact Lie group $I(M)$ (the group of all isometries of M endowed with the topology of uniform convergence). But now Theorem D applies to the action of $I(M)$ on M implying that (R, M, π_1) can be embedded in a linear flow (where the transition functions are in fact orthogonal transformations) on some Euclidean space R^n. This accomplishes the proof of our theorem, since every linear flow on R^n can be obtained as the flow generated by the equation $\dot{x} = Ax$ where A is a constant matrix.

As a simple example let X be the subset of the plane R^2 with the open disc removed, i.e., in polar coordinates $X = \{(r, \phi): r \geqslant 1\}$ and let (R, X, π) be the flow on X defined by the autonomous system

$$dr/dt = f(r), \qquad d\phi/dt = 1$$

where f is a function continuous and decreasing on $[1, \infty)$, negative on $(1, \infty)$ and satisfying $f(1) = 0$.

The trajectories of this flow spiral around the unit circle C and approach it as $t \to \infty$; thus C is evidently the manifold which is globally and asymptotically stable. The nonexpansiveness of π follows from negativeness of f. Thus, the flow (R, X, π) satisfies the conditions of our theorem. We observe that X/C is homeomorphic to R^2 so that the flow $(R, X/C, \pi^*)$ can be linearized in R^4 and the flow (R, C, π_1) which is a rotation can of course be represented in R^2. (In [4] it is shown, see Corollary 9 of this paper, that in this particular case the flow $(R, X/C, \pi^*)$ can be linearized also in R^2.)

REMARK. The treatment of the linearization problem as we offer it in this note is of highly nonconstructive character. Also, we are so far unable to produce any a priori estimate of the dimension n of the linearization space R^n from the knowledge of the flow (R, X, π) and the dimension of its underlying space X.

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