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A COMMON FIXED POINT THEOREM FOR COMMUTING MAPPINGS

SHIH-SEN CHANG

ABSTRACT. Results generalizing and unifying fixed point theorems of Jungck, Ciric, Das, Naik, Rhoades and Leader are established.

1. Introduction. Recently, some fixed point and common fixed point theorems for commuting mappings in complete metric spaces were proved by Jungck [1]–[3], Das, Naik [4], etc. The purpose of this paper is to generalize and unify some of these results.

2. Preliminaries. Throughout this paper let (X, d) be a complete metric space, f a self-mapping on X such that for some positive integer m f^m is continuous. Let g_i: $f^{m-1}(X) \to X$, $i = 1, 2, \ldots,$ be a sequence of mappings which commutes with f and satisfies

$$g_i(f^{m-1}(X)) \subset f^m(X), \qquad i = 1, 2, \ldots. \tag{2.1}$$

Further let there exist a sequence of positive integers $\{m_i\}_{i=1}^{\infty}$ such that for any positive integers i, j and $x, y \in f^{m-1}(X)$ the following holds

$$\begin{aligned} &d(g_i^{m_i}(x), g_j^{m_j}(y)) \\ &\leqslant A\big(\max\{d(f(x), f(y)), d(f(x), g_i^{m_i}(x)), d(f(y), g_j^{m_j}(y)), \\ &\qquad d(f(y), g_i^{m_i}(x)), d(f(x), g_j^{m_j}(y))\}\big), \end{aligned} \tag{2.2}$$

where the function A satisfies the following conditions (A1), (A2) or (A1), (A3):

(A1). A: $[0, \infty) \to [0, \infty)$ is nondecreasing and right continuous.

(A2). For any real number $q \in [0, \infty)$ there exists a suitable real number $t(q) \in [0, \infty)$ such that

(a) $t(q)$ is the upper bound of the set

$$\{t \in [0, \infty): t \leqslant q + A(t)\},$$

(b) $\lim_{n\to\infty} A^n(t(q)) = 0$.

(A3). For every $t > 0$, $A(t) < t$ and

$$\lim_{t\to\infty} (t - A(t)) = \infty.$$

It is easy to see that from (2.1) we have

$$\begin{aligned} g_i^{m_i} f^{m-1}(X) &= g_i^{m_i-1} g_i f^{m-1}(X) \subset g_i^{m_i-1}(f^m(X)) \\ &= g_i^{m_i-1} f^{m-1} f(X) \subset g_i^{m_i-1} f^{m-1}(X) \subset g_i^{m_i-2} f^m(X). \end{aligned}$$

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Repeating this procedure it follows that

$$g_i^{m_i}(f^{m-1}(X)) \subset f^m(X), \qquad i = 1, 2, \ldots. \tag{2.3}$$

Under the assumptions stated above, now we define a sequence of points $\{x_n\}$ as follows.

For any $x_1 \in f^{m-1}(X)$ let $x_2 \in f^{m-1}(X)$, guaranteed by (2.3), be such that $g_1^{m_1}(x_1) = f(x_2)$. Having defined $x_n \in f^{m-1}(X)$, let $x_{n+1} \in f^{m-1}(X)$ be such that

$$g_n^{m_n}(x_n) = f(x_{n+1}), \qquad n = 1, 2, \ldots.$$

Let

$$y_n = g_n^{m_n}(x_n) = f(x_{n+1}), \qquad n = 1, 2, \ldots. \tag{2.4}$$

Now we prove the following lemma.

LEMMA 1. *Let A satisfy conditions* (A1) *and* (A2). *Then the sequence $\{y_n\}_{n=1}^{\infty}$ defined by* (2.4) *is a Cauchy sequence in X.*

PROOF. Indeed, for any positive integers i, j we have

$$\begin{aligned} d(y_i, y_j) &= d(g_i^{m_i}(x_i), g_j^{m_j}(x_j)) \\ &\leqslant A\big(\max\{d(f(x_i), f(x_j)), d(f(x_i), g_i^{m_i}(x_i)), d(f(x_j), g_j^{m_j}(x_j)), \\ &\qquad d(f(x_j), g_i^{m_i}(x_i)), d(f(x_i), g_j^{m_j}(x_j))\}\big) \\ &\leqslant A\big(\max\{d(y_{i-1}, y_{j-1}), d(y_{i-1}, y_i), d(y_{j-1}, y_j), \\ &\qquad d(y_{j-1}, y_i), d(y_{i-1}, y_j)\}\big). \end{aligned}$$

Thus by the assumption (A1), for any positive integers m, n $(m < n)$, we have

$$\sup_{m<i,j<n} d(y_i, y_j) \leqslant A\Big(\sup_{m-1<i,j<n} d(y_i, y_j)\Big). \tag{2.5}$$

However

$$\sup_{1<i,j<n} d(y_i, y_j) \leqslant d(y_1, y_2) + \sup_{2<i,j<n} d(y_i, y_j). \tag{2.6}$$

Taking $m = 2$ in (2.5), it follows from (2.6) that

$$\sup_{1<i,j<n} d(y_i, y_j) \leqslant d(y_1, y_2) + A\Big(\sup_{1<i,j<n} d(y_i, y_j)\Big). \tag{2.7}$$

Letting the real number $q = d(y_1, y_2)$ and using condition (A2) it follows from (2.7) that there exists a real number $t(q) \in [0, \infty)$ such that

$$\sup_{1<i,j<n} d(y_i, y_j) \leqslant t(q). \tag{2.8}$$

Take $m = 2$ in (2.5) again, and consider (2.8). Then we have

$$\sup_{2<i,j<n} d(y_i, y_j) \leqslant A(t(q)). \tag{2.9}$$

Taking $m = 3$ in (2.5) and using (2.9) it follows that

$$\sup_{3<i,j<n} d(y_i, y_j) \leqslant A^2(t(q)). \tag{2.10}$$

Repeating this procedure, for any positive integers m, n ($m < n$) we obtain

$$\sup_{m<i,j<n} d(y_i, y_j) \leqslant A^{m-1}(t(q)). \tag{2.11}$$

Letting $m \to \infty$ (hence $n \to \infty$) in (2.11), from condition (A2) we obtain

$$0 \leqslant \lim_{m\to\infty} \sup_{m<i,j<n} d(y_i, y_j) \leqslant \lim_{m\to\infty} A^{m-1}(t(q)) = 0.$$

This shows that $\{y_i\}$ is a Cauchy sequence in X.

LEMMA 2. *Let A satisfy the conditions* (A1) *and* (A3). *Then*

(i) *for every $t > 0$ we have*

$$\lim_{n\to\infty} A^n(t) = 0;$$

(ii) *for any sequence $\{t_n\}$ of nonnegative real numbers satisfying the following condition*

$$t_{n+1} \leqslant A(t_n), \qquad n = 1, 2, \ldots,$$

we have $\lim_{n\to\infty} t_n = 0$.

REMARK 1. Lemma 2(i) was essentially obtained by Singh and Meade [5]. We include the proof for completeness.

PROOF OF LEMMA 2. (i) For every $t > 0$, (A3) yields $A(t) < t$. Repeating this procedure, we obtain

$$A^n(t) \leqslant A^{n-1}(t) \leqslant \cdots \leqslant A(t) < t.$$

By the right continuity of A it follows that

$$\lim_{n\to\infty} A^n(t) = \lim_{n\to\infty} A(A^{n-1}(t)) = A\Big(\lim_{n\to\infty} A^{n-1}(t)\Big).$$

Put $v = \lim_{n\to\infty} A^n(t)$. From the preceding relation we obtain $v = A(v)$. If $v \neq 0$, then it follows from (A3) that $v = A(v) < v$. This is a contradiction. Hence $v = 0$, i.e. $\lim_{n\to\infty} A^n(t) = 0$.

(ii) From the condition of Lemma 2, it is easy to see

$$t_{n+1} \leqslant A(t_n) \leqslant A^2(t_{n-1}) \leqslant \cdots \leqslant A^n(t_1).$$

Letting $n \to \infty$ and using the conclusion (i) of Lemma 2, we obtain

$$0 \leqslant \lim_{n\to\infty} t_{n+1} \leqslant \lim_{n\to\infty} A^n(t_1) = 0.$$

This completes the proof of Lemma 2.

From Lemma 2 we can prove the following lemma.

LEMMA 3. *Let A satisfy conditions* (A1) *and* (A3). *Then the sequence $\{y_n\}_{n=1}^{\infty}$ defined by* (2.4) *is also a Cauchy sequence in X.*

PROOF. By the preceding inequality (2.7) we can prove that

$$\lim_{n\to\infty} \sup_{1<i,j<n} d(y_i, y_j) = \sup_{i,j>1} d(y_i, y_j) < \infty.$$

Suppose this is not the case, therefore

$$\lim_{n\to\infty} \sup_{1\leq i,j\leq n} d(y_i, y_j) = \infty.$$

It follows from (A3) and (2.7) that

$$\infty = \lim_{n\to\infty}\left(\sup_{1\leq i,j\leq n} d(y_i, y_j) - A\left(\sup_{1\leq i,j\leq n} d(y_i, y_j)\right)\right) \leq d(y_1, y_2).$$

This is a contradiction. This contradiction implies that

$$\lim_{n\to\infty} \sup_{1\leq i,j\leq n} d(y_i, y_j) = \sup_{i,j\geq 1} d(y_i, y_j) < \infty.$$

Now we define a decreasing sequence of nonnegative real numbers

$$t_m = \sup_{i,j\geq m} d(y_i, y_j), \qquad m = 1, 2, \ldots. \tag{2.11}$$

From (2.5), we obtain

$$t_m \leq A(t_{m-1}), \qquad m = 2, 3, \ldots.$$

By Lemma 2(ii) this shows $t_m \to 0$ $(m \to \infty)$, i.e.

$$\lim_{m\to\infty} \sup_{i,j\geq m} d(y_i, y_j) = 0.$$

Thus we have proved $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

This completes the proof of Lemma 3.

3. Main results. We now formulate the main results of this paper as follows:

THEOREM 1. *Let (X, d) be a complete metric space, f a self-mapping on X such that for some positive integer m f^m is continuous. Let $\{g_i\}_{i=1}^{\infty}$: $f^{m-1}(X) \to X$ be a sequence of mappings such that* (2.1) *is satisfied, and suppose g_i commutes with f, $i = 1, 2, \ldots$. Further suppose there exists a sequence of positive integers $\{m_i\}_{i=1}^{\infty}$ such that for any positive integers i, j and any $x, y \in f^{m-1}(X)$* (2.2) *holds, where the function A appearing in* (2.2) *satisfies the conditions* (A1), (A2) *or* (A1), (A3). *Then f and g_i, $i = 1, 2, \ldots$, have a unique common fixed point $f(y_*)$, where $y_* \in X$ is the limit of the sequence $\{y_n\}_{n=1}^{\infty}$ defined by* (2.4).

PROOF. Since A satisfies conditions (A1), (A2) (or (A1), (A3)), from Lemma 1 (or Lemma 3) the sequence $\{y_n\}_{i=1}^{\infty}$ defined by (2.4) is a Cauchy sequence in X. Suppose it converges to some point y_* in X. By the continuity of f^m, $\{f^m(y_*)\}$ converges to $f^m(y_*)$.

However by (2.4)

$$g_n^{m_n}(f^{m-1}(y_{n-1})) = g_n^{m_n}(f^m(x_n)) = f^m(g_n^{m_n}(x_n)) = f^m(y_n), \tag{3.1}$$

and this implies

$$\lim_{n\to\infty} g_n^{m_n}(f^{m-1}(y_{n-1})) = \lim_{n\to\infty} f^m(y_n) = f^m(y_*). \tag{3.2}$$

Further, for any positive integer i, $i = 1, 2, \ldots,$ we obtain from (3.1) that

$$\begin{aligned} d\Big(f^m(y_n), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big) &= d\Big(g_n^{m_n}\big(f^{m-1}(y_{n-1})\big), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big) \\ &\leqslant A\Big(\max\Big\{d\big(f^m(y_{n-1}), f^m(y_*)\big), d\big(f^m(y_{n-1}), g_n^{m_n}(f^{m-1}(y_{n-1}))\big), \\ &\qquad d\Big(f^m(y_*), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big), d\big(f^m(y_*), g_n^{m_n}(f^{m-1}(y_{n-1}))\big), \\ &\qquad d\Big(f^m(y_{n-1}), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big)\Big\}\Big) \\ &\leqslant A\Big(\max\Big\{d\big(f^m(y_{n-1}), f^m(y_*)\big), d\big(f^m(y_{n-1}), f^m(y_n)\big), \\ &\qquad d\Big(f^m(y_*), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big), d\big(f^m(y_*), f^m(y_n)\big), \\ &\qquad d\Big(f^m(y_{n-1}), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big)\Big\}\Big). \end{aligned} \tag{3.3}$$

Since

$$\lim_{n\to\infty} f^m(y_n) = f^m(y_*),$$

and

$$\lim_{n\to\infty} g_n^{m_n}\big(f^{m-1}(y_{n-1})\big) = f^m(y_*),$$

then for any $\varepsilon > 0$ there exists a positive integer n_0 such that for $n \geqslant n_0$ we have

$$d\big(f^m(y_{n-1}), f^m(y_*)\big) < \frac{\varepsilon}{2},$$

$$d\big(g_n^{m_n}(f^{m-1}(y_{n-1})), f^m(y_*)\big) < \frac{\varepsilon}{2}.$$

Hence, for $n \geqslant n_0$, we have

$$\begin{aligned} &d\Big(f^m(y_n), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big) \\ &\qquad \leqslant A\Big(\max\Big\{\frac{\varepsilon}{2}, \varepsilon, d\Big(f^m(y_*), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big), \frac{\varepsilon}{2}, \\ &\qquad\qquad \frac{\varepsilon}{2} + d\Big(f^m(y_*), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big)\Big\}\Big) \\ &\qquad \leqslant A\Big(\varepsilon + d\Big(f^m(y_*), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big)\Big). \end{aligned} \tag{3.4}$$

First, letting $n \to \infty$ in the left side of (3.4) it follows that

$$d\Big(f^m(y_*), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big) \leqslant A\Big(\varepsilon + d\Big(f^m(y_*), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big)\Big).$$

Next letting $\varepsilon \searrow 0$ in the right side of the preceding inequality and invoking the right continuity of A we have

$$d\Big(f^m(y_*), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big) \leqslant A\Big(d\Big(f^m(y_*), g_i^{m_i}\big(f^{m-1}(y_*)\big)\Big)\Big).$$

By (A3) this shows

$$f^m(y_*) = g_i^{m_i}\big(f^{m-1}(y_*)\big), \qquad i = 1, 2, \ldots. \tag{3.5}$$

Therefore from (3.5) we obtain

$$\begin{aligned} d\left(g_i^{m_i}\left(f^m(y_*)\right), f^m(y_*)\right) &= d\left(g_i^{m_i}\left(f^m(y_*), g_j^{m_j}\left(f^{m-1}(y_*)\right)\right)\right) \\ &\leqslant A(\max\{d\left(f^{m+1}(y_*), f^m(y_*)\right), d\left(f^{m+1}(y_*), f^{m+1}(y_*)\right), \\ &\qquad d\left(f^m(y_*), f^m(y_*)\right), d\left(f^m(y_*), g_i^{m_i} f^m(y_*)\right), \\ &\qquad\qquad d\left(f^{m+1}(y_*), f^m(y_*)\right)\}) \\ &= A\left(d\left(g_i^{m_i} f^m(y_*), f^m(y_*)\right)\right). \end{aligned}$$

By (A3), this implies that

$$g_i^{m_i}\left(f^m(y_*)\right) = f^m(y_*), \qquad i = 1, 2, \ldots. \tag{3.6}$$

Now we prove $f^m(y_*)$ is the unique common fixed point of f and g_i, $i = 1, 2, \ldots$.

Indeed, in view of (3.5) and (3.6) we have

$$f\left(f^m(y_*)\right) = f\left(g_i^{m_i}\left(f^{m-1}(y_*)\right)\right) = g_i^{m_i} f^m(y_*) = f^m(y_*). \tag{3.7}$$

On the other hand, for any i, $i = 1, 2, \ldots,$ from (3.6) and (3.7) we have

$$\begin{aligned} d\left(f^m(y_*), g_i f^m(y_*)\right) &= d\left(g_i^{m_i} f^m(y_*), g_i^{m_i} g_i\left(f^m(y_*)\right)\right) \\ &\leqslant A\Big(\max\{d\left(f^{m+1}(y_*), f\left(g_i f^m(y_*)\right)\right), d\left(f^{m+1}(y_*), f^m(y_*)\right), \\ &\qquad d\left(f g_i f^m(y_*), g_i f^m(y_*)\right), d\left(f g_i f^m(y_*), f^m(y_*)\right), \\ &\qquad\qquad d\left(f^{m+1}(y_*), g_i f^m(y_*)\right)\}\Big) \\ &= A\Big(\max\{d\left(f^m(y_*), g_i f^m(y_*)\right), 0, 0, \\ &\qquad d\left(g_i f^m(y_*), f^m(y_*)\right), d\left(f^m(y_*), g_i f^m(y_*)\right)\}\Big) \\ &= A\left(d\left(f^m(y_*), g_i f^m(y_*)\right)\right). \end{aligned} \tag{3.8}$$

By (A3) from (3.8) we have

$$f^m(y_*) = g_i f^m(y_*), \qquad i = 1, 2, \ldots. \tag{3.9}$$

Combining (3.7) with (3.9), we obtain $f^m(y_*)$ is a common fixed point of f and g_i, $i = 1, 2, \ldots$.

To prove $f^m(y_*)$ is the unique common fixed point of f and g_i, we proceed as follows.

Suppose there exists another common fixed point x_* of f and g_i, $i = 1, 2, \ldots,$ such that

$$g_i x_* = x_*, \quad i = 1, 2, \ldots, \qquad f x_* = x_*.$$

Consequently

$$\begin{aligned} d\big(x_*, f^m(y_*)\big) &= d\big(g_i^{m_i}(x_*), g_j^{m_j} f^m(y_*)\big) \\ &\leqslant A\Big(\max\Big\{ d\big(f(x_*), f^{m+1}(y_*)\big), d\big(f(x_*), g_i^{m_i}(x_*)\big), \\ &\qquad d\big(f^{m+1}(y_*), g_j^{m_j} f^m(y_*)\big), \\ &\qquad\qquad d\big(f^{m+1}(y_*), g_i^{m_i}(x_*)\big), d\big(f(x_*), g_j^{m_j}\big(f^m(y_*)\big)\big)\Big\}\Big) \\ &= A\Big(\max\Big\{ d\big(x_*, f^m(y_*)\big), 0, 0, d\big(f^m(y_*), x_*\big), d\big(x_*, f^m(y_*)\big)\Big\}\Big) \\ &= A\Big(d\big(x_*, f^m(x_*)\big)\Big). \end{aligned}$$

It implies $x_* = f^m(y_*)$.

This completes the proof of Theorem 1.

From Theorem 1 we can easily deduce the following corollary.

COROLLARY. *Let (X, d) be a complete metric space, f a self-mapping on X such that for some positive integer m, f^m is continuous. Let $\{g_i\}_{i=1}^{\infty}: f^{m-1}(X) \to X$ be a sequence of mappings such that* (2.1) *is satisfied, and suppose g_i commutes with f, $i = 1, 2, \ldots$. Further suppose there exists a sequence of positive integers $\{m_i\}_{i=1}^{\infty}$ such that for any positive integer i, j and any $x, y \in f^{m-1}(X)$ the following holds*:

$$\begin{aligned} d\big(g_i^{m_i}(x), g_j^{m_j}(y)\big) \leqslant k \max\Big\{ & d(f(x), f(y)), d(f(x), g_i^{m_i}(x)), d\big(f(y), g_j^{m_j}(y)\big), \\ & d(f(y), g_i^{m_i}(x)), d\big(f(x), g_j^{m_j}(y)\big)\Big\}, \end{aligned}$$

where k is a constant: $0 < k < 1$.

Then the conclusion of Theorem 1 *still holds.*

PROOF. Taking $A(t) = kt$, $t \geqslant 0$, it is easy to see that $A(t)$ satisfies the assumptions (A1), (A2), (A3). Therefore the conclusion of the Corollary follows from the theorem.

REMARK 2. The results of Das and Naik [**4**, Theorem 2.1, Theorem 3.1, Theorem 3.2 and Theorem 4.1], Ciric [**6**, Theorem 1 and Theorem 2], Jungck [**3**], Rhoades [**7**, Theorem 23] are all special cases of our Corollary.

In a sense, the theorem in this paper is also a generalization and improvement of Cheh-chih Yen [**8**], Murakami and Cheh-chih Yen [**9**] and Leader [**10**].

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Department of Mathematics, Sichuan University, Chengdu, Sichuan, 610064, People's Republic of China