STIEFEL-WHITNEY CLASSES IN $H^* BO<\phi(r)>$

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Abstract. We determine the Stiefel-Whitney classes in $H^*(BO; \mathbb{Z}_2)$ which are mapped nontrivially by the homomorphism induced by the covering projection $p$: $BO<\phi(r)>ightarrow BO$.

Let $\phi(r)$ be the function defined by $\phi(r) = 8a + 2b$ where $r = 4a + b$ with $0 < b < 3$ and let $BO<\phi(r)>$ be the $(\phi(r) - 1)$-connected covering of $BO$. It follows from Stong’s computation of $H^*(BO<\phi(r)>; \mathbb{Z}_2)$ in [4] that the covering map

$p: BO<\phi(r)>ightarrow BO$

maps the Stiefel-Whitney classes $w_i \in H^*(BO; \mathbb{Z}_2)$ to generators in $\pi_*BO<\phi(r)>$. The remaining classes are mapped to decomposables. In this note we determine which Stiefel-Whitney classes are mapped to nonzero decomposables. In doing so we display a relationship between $H^*BO<\phi(r)>$ and the cohomology of certain spaces related to $RP^\infty$.

Let $A$ denote the mod 2 Steenrod algebra and $A_r$ the subalgebra generated by $Sq^1, Sq^2, Sq^4, \ldots, Sq^{2^r}$. Let $P = \mathbb{Z}_2[x, x^{-1}]$ be the ring of Laurent polynomials in one variable $x$ of degree +1, made into a module over $A$ by setting

$Sq^i x^j = \frac{j(j - 1) \cdots (j - i + 1)}{1 \cdot 2 \cdots i} x^{i+j}$.

Let $F_{-2r}$ denote the $A_r$-submodule of $P$ generated by $x^j$ with $j < -2$.

**Theorem A.** The class $p^*w_n$ is nonzero in $H^*BO<\phi(r)>$ if and only if $\Sigma P/F_{-2r}$ is nonzero in dimension $n$. The Poincaré series for $\Sigma P/F_{-2r}$ is

$$\frac{1}{1 - \sum_{i=1}^{\infty} (1 + i^r)(1 + i^{3^{i-1}}) \cdots (1 + i^{(2^{i-1})^2})(1 + i^{2^{i-1}})}.$$

The theorem is a consequence of the following two lemmas and the fact (from [2]) that as $\mathbb{Z}_2$-vector spaces

$$\Sigma P/F_{-2r} \cong \bigoplus_{j=0}^{2^r} \Sigma^j(A_r \otimes_{A_{r-1}} \mathbb{Z}_2).$$

**Lemma 1.** The class $p^*w_n$ is nonzero in $H^*BO<\phi(r)>$ if and only if $A \otimes_{A_{r-1}} \mathbb{Z}_2$ is nonzero in dimension $n$.

**Lemma 2.** $A \otimes_{A_{r-1}} \mathbb{Z}_2$ and $\bigoplus_{j=0}^{2^r} \Sigma^j(A_r \otimes_{A_{r-1}} \mathbb{Z}_2)$ are nonzero in exactly the same dimensions.

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Proof of Lemma 1. Lemma 1 follows as a corollary to the following theorem which we prove by using a slight generalization of an argument of Giambalvo [1].

Theorem B. The map \( e : A \otimes_{A,\pi} \mathbb{Z}_2 \to H^* \text{MO} \langle \phi(r) \rangle \) given by evaluation on the Thom class \( U \in H^* \text{MO} \langle \phi(r) \rangle \) is a monomorphism.

As remarked in [1] it suffices to prove that \( e \) is a monomorphism on the primitive elements of \( A \otimes_{A,\pi} \mathbb{Z}_2 \). Since \( \chi(A \otimes_{A,\pi} \mathbb{Z}_2)^* \approx \mathbb{Z}_2[\xi_1^2, \xi_2^{2^{-1}}, \ldots, \xi_r^2, \ldots] \), where \( \xi_i \) is the Milnor basis element of \( A^* \) in dimension \( 2^i - 1 \); there are primitives in \( A \otimes_{A,\pi} \mathbb{Z}_2 \) only in degrees \( 2^i, 3 \cdot 2^{i-1}, 7 \cdot 2^{i-2}, \ldots, 2^{r+1} - 2 \) and \( 2^i - 1 \) for \( i > r + 1 \). For purely dimensional reasons the first \( r + 1 \) primitives must be \( \text{Sq}^2, \text{Sq}^3 2^{-1}, \ldots, \text{Sq}^{r+1} 2^{-2} \). The remaining primitives \( \text{Q}^{2^i - 1}, i > r + 1 \), are projections of primitives in \( A \). Now

\[
\text{Sq}^j U = w_j \cdot U \quad \text{for } j = 2^i, 3 \cdot 2^{i-1}, \ldots, 2^{r+1} - 2
\]

and

\[
\text{Q}_1^{2^i-1} U = w_{2^i-1} \cdot U + (\text{decomposables}) \cdot U \quad \text{for } i > r + 1.
\]

Since the numbers \( j - 1 \) and \( 2^j - 2 \) for \( i > r + 1 \) all have at least \( r \) ones in their dyadic expansion, \( \text{Sq}^j U \) and \( \text{Q}_1^{2^i-1} U \) are nonzero by Stong's result, proving that \( e \) is a monomorphism. To deduce the lemma we need to show that \( \text{SQ}^n \) is nonzero in \( A \otimes_{A,\pi} \mathbb{Z}_2 \) if and only if there is a monomial of dimension \( n \) in \( \chi(A \otimes_{A,\pi} \mathbb{Z}_2)^* \). To see this recall that there is a right \( A \)-module structure on \( A^* \), given by the duality pairing, with the property that

\[
\xi_k \chi \text{Sq}^j = \begin{cases} 
\xi_j & \text{if } j = 2^k - 2^r, \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( \chi \) commutes with the diagonal homomorphism in \( A \), we have

\[
\langle (\xi_1^2, \xi_2^2, \ldots, \xi_r^2), \chi \text{Sq}^n \rangle = 1 \quad \text{if } n = e_1 + 3e_2 + \cdots + (2^r - 1)e_r.
\]

Considering the induced right \( \chi(A \otimes_{A,\pi} \mathbb{Z}_2)^* \)-module structure on \( \chi(A \otimes_{A,\pi} \mathbb{Z}_2)^* \) then yields the result that \( \chi \text{Sq}^n \) is nonzero in \( \chi(A \otimes_{A,\pi} \mathbb{Z}_2)^* \) if and only if it is a monomial of dimension \( n \) in \( \chi(A \otimes_{A,\pi} \mathbb{Z}_2)^* \) completing the proof of Lemma 1.

Proof of Lemma 2. By only a slight modification of the argument given by Peterson, in [3], to compute \( \chi(A \otimes A, \mathbb{Z}_2)^* \) and the fact that

\[
A^*_r = A^*/(\xi_1^{2^r+1}, \xi_2^2, \ldots, \xi_{r+1}, \xi_{r+2}, \ldots),
\]

we can show that as \( \mathbb{Z}_2 \)-vector spaces

\[
\chi(A \otimes_{A,\pi} \mathbb{Z}_2)^* \approx \Lambda(\xi_1^2, \xi_2^{2^{-1}}, \ldots, \xi_r^2, \xi_r + 1)
\]

where the right-hand side is the exterior algebra over \( \mathbb{Z}_2 \) generated by \( \xi_1^2, \ldots, \xi_{r+1} \).

We define a map \( \lambda : \Sigma^{2^r+1} \chi(A \otimes_{A,\pi} \mathbb{Z}_2) \to \chi(A \otimes_{A,\pi} \mathbb{Z}_2)^* \) of vector spaces over \( \mathbb{Z}_2 \), by

\[
\lambda(\xi_1^{2^r} \xi_2^{2^{-1} \cdot k_2} \cdots \xi_{r+1}^{e_{r+1}}) = \xi_1^{2^{e_1+2e_2}} \xi_2^{2^{-1} \cdot k_2} \cdots \xi_{r+1}^{e_{r+1}}
\]
for $e_i = 0$ or 1. Since $\lambda$ is a monomorphism it follows that $A \otimes A_{-1} Z_2$ is nonzero in dimension $n$ if $\bigoplus_{j=0}^{(2^{r+1})} \Sigma^j A_r \otimes A_{-1} Z_2$ is. Conversely we define a map

$$\rho: \chi(A \otimes A_{-1} Z_2)^* \rightarrow \bigoplus_{j=0}^{(2^{r+1})} \Sigma^j \chi(A_r \otimes A_{-1} Z_2)^*$$

by the following procedure. Suppose that

$$\xi_1^{2a_1} \xi_2^{2a_2} \cdots \xi_{r+1}^{a_{r+1}} \cdots \xi_j^a$$

is a monomial in $\chi(A \otimes A_{-1} Z_2)^*$ of dimension $n$. Begin by replacing it with the monomial

$$(3) \xi_1^{2a_1 + 2^{-r}b_1} \xi_2^{2^{-r}b_2} \cdots \xi_{r+1}^{b_{r+1}} \cdots \xi_j^a$$

where $\omega = \sum_{i=r+1}^j a_i$ and $v = a_{r+2} + 3a_{r+3} + 7a_{r+4} + \cdots + (2^{j-r-1} - 1)a_j$. This monomial is also of dimension $n$. Next we inductively “reduce” the monomial while at the same time preserving its dimension. If (3) is of the form

$$\xi_1^{2a_1 + 2^{-i}b_1} \cdots \xi_{i-1}^{2^{-i-1}b_{i-1}} \xi_i^{2^{-i-1}b_i} \cdots \xi_{r+1}^{b_{r+1}} \cdots \xi_j^a$$

with $i < r + 1$, $b_i \in Z$ and each $e_i$ either 0 or 1 we replace it with the monomial

$$(4) \xi_1^{2(b_1 + c_1 + 2^{-i-1}b_{i+1})} \cdots \xi_{i-1}^{2^{-i-1}b_{i-1}} \cdots \xi_{r+1}^{b_{r+1}} \cdots \xi_j^a$$

where $b_i = e_i + 2e_i-1 + 4e_i-2 + \cdots + 2^{i+2} - 2i + 2^{i-1} - 1$, each $e_i$ is either 0 or 1 and $c = e_i-1 + 3e_i-2 + \cdots + (2^{i-2} - 1)e_2 + (2^{i-1} - 1)t$. The monomial (4) also has dimension $n$. Continuing in this fashion we end up with a monomial of the form

$$(5) \xi_1^{2^{2 i+1} + 2^{-i}b_1} \cdots \xi_{i-1}^{2^{-i}b_{i-1}} \cdots \xi_{r+1}^{b_{r+1}} \cdots \xi_j^a$$

with each $e_j$ zero or one. We define $\rho(\xi_1^{2a_1} \xi_2^{2a_2} \cdots \xi_j^a)$ to be the nonzero monomial $\xi_1^{2a_1} \cdots \xi_{r+1}^{\xi_j^a}$ in $\Sigma^{2^{r+1}} \chi(A_r \otimes A_{-1} Z_2)^*$ of dimension $n$; so

$$\bigoplus_{j=0}^{(2^{r+1})} \Sigma^j \chi(A_r \otimes A_{-1} Z_2)^*$$

is nonzero in dimension $n$ completing the proof of Lemma 2.

**ADDED IN PROOF.** Related results about the vanishing of Stiefel-Whitney classes have been proved by R. Stong. See §3 of *Cobordism and Stiefel-Whitney Numbers*, Topology 4 (1965), 241–246.

**REFERENCES**

4. R. Stong, *Determination of $H^*(BO(k, \ldots, \infty))$ and $H^*(BU(k, \ldots, \infty))$, Trans. Amer. Math. Soc. 104 (1963), 526–544.

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