STIEFEL-WHITNEY CLASSES IN $H^* BO\langle \phi(r) \rangle$

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Abstract. We determine the Stiefel-Whitney classes in $H^*(BO; \mathbb{Z}_2)$ which are mapped nontrivially by the homomorphism induced by the covering projection $p: BO\langle \phi(r) \rangle \to BO$.

Let $\phi(r)$ be the function defined by $\phi(r) = 8a + 2b$ where $r = 4a + b$ with $0 < b < 3$ and let $BO\langle \phi(r) \rangle$ be the $(\phi(r) - 1)$-connected covering of $BO$. It follows from Stong's computation of $H^*(BO\langle \phi(r) \rangle; \mathbb{Z}_2)$ in [4] that the covering map $p: BO\langle \phi(r) \rangle \to BO$ maps the Stiefel-Whitney classes $w_i \in H^*(BO; \mathbb{Z}_2)$ to generators in $\pi^*(BO\langle \phi(r) \rangle)$ if $i - 1$ has at least $r$ ones in its dyadic expansion. The remaining classes are mapped to decomposables. In this note we determine which Stiefel-Whitney classes are mapped to nonzero decomposables. In doing so we display a relationship between $H^* BO\langle \phi(r) \rangle$ and the cohomology of certain spaces related to $RP^\infty$.

Let $A$ denote the mod 2 Steenrod algebra and $A_r$ the subalgebra generated by $Sq^1, Sq^2, Sq^4, \ldots, Sq^{2^r}$. Let $P = \mathbb{Z}_2[x, x^{-1}]$ be the ring of Laurent polynomials in one variable $x$ of degree $+1$, made into a module over $A$ by setting $Sq^ix^j = \frac{j(j - 1) \cdots (j - i + 1)}{1 \cdot 2 \cdots i} x^{i+j}$.

Let $F_{-2r}$ denote the $A_r$-submodule of $P$ generated by $x^j$ with $j < -2$.

Theorem A. The class $p^* w_n$ is nonzero in $H^* BO\langle \phi(r) \rangle$ if and only if $\Sigma P/F_{-2r}$ is nonzero in dimension $n$. The Poincaré series for $\Sigma P/F_{-2r}$ is

$$
\frac{1}{1 - t^{2^r + 1}}(1 + t^2)(1 + t^{3^{2^r - 1}}) \cdots (1 + t^{(2^r - 1)2})(1 + t^{2^{r+1} - 1}).
$$

The theorem is a consequence of the following two lemmas and the fact (from [2]) that as $\mathbb{Z}_2$-vector spaces

$$
\Sigma P/F_{-2r} \cong \bigoplus_{j=0}^{2^{r+1}} \Sigma^j(A_r \otimes_{A_r} \mathbb{Z}_2).
$$

Lemma 1. The class $p^* w_n$ is nonzero in $H^* BO\langle \phi(r) \rangle$ if and only if $A \otimes_{A_r} \mathbb{Z}_2$ is nonzero in dimension $n$.

Lemma 2. $A \otimes_{A_r} \mathbb{Z}_2$ and $\bigoplus_{j=0}^{2^{r+1}} \Sigma^j(A_r \otimes_{A_r} \mathbb{Z}_2)$ are nonzero in exactly the same dimensions.
Proof of Lemma 1. Lemma 1 follows as a corollary to the following theorem which we prove by using a slight generalization of an argument of Giambalvo [1].

Theorem B. The map \( e: A \otimes A_{(r)}Z_2 \rightarrow H^*MO(\phi(r)) \) given by evaluation on the Thom class \( U \in H^*MO(\phi(r)) \) is a monomorphism.

As remarked in [1] it suffices to prove that \( e \) is a monomorphism on the primitive elements of \( A \otimes A_{(r)}Z_2 \). Since \( \chi(A \otimes A_{(r)}Z_2)^* \cong Z_2[\xi^2, \xi^2, ..., \xi_r, ...,] \), where \( \xi_r \) is the Milnor basis element of \( A^* \) in dimension \( 2^r - 1 \); there are primitives in \( A \otimes A_{(r)}Z_2 \) only in degrees \( 2^r, 3 \cdot 2^r - 1, 7 \cdot 2^r - 2, ..., 2^r + 1 - 2 \) and \( 2^r - 1 \) for \( i > r + 1 \). For purely dimensional reasons the first \( r + 1 \) primitives must be \( Sq^2, Sq^3, ..., Sq^{2^r - 1} \). The remaining primitives \( Q^{2^r - 1}, i > r + 1 \), are projections of primitives in \( A \). Now

\[
Sq^jU = w_j \cdot U \quad \text{for } j = 2^r, 3 \cdot 2^r - 1, ..., 2^r + 1 - 2
\]

and

\[
Q^{2^r - 1}U = w_{2^r - 1} \cdot U + (\text{decomposables}) \cdot U \quad \text{for } i > r + 1.
\]

Since the numbers \( j - 1 \) and \( 2^r - 2 \) for \( i > r + 1 \) all have at least \( r \) ones in their dyadic expansion, \( Sq^jU \) and \( Q^{2^r - 1}U \) are nonzero by Stong's result, proving that \( e \) is a monomorphism. To deduce the lemma we need to show that \( Sq^n \) is nonzero in \( A \otimes A_{(r)}Z_2 \) if and only if there is a monomial of dimension \( n \) in \( \chi(A \otimes A_{(r)}Z_2)^* \). To see this recall that there is a right \( A \)-module structure on \( A^* \), given by the duality pairing, with the property that

\[
\xi_k \chi Sq^j = \begin{cases} \xi_j & \text{if } j = 2^k - 2^r, \\ 0 & \text{otherwise.} \end{cases}
\]

Since \( \chi \) commutes with the diagonal homomorphism in \( A \), we have

\[
\langle (\xi_1^j, \xi_2^j, ..., \xi_r^j), \chi Sq^n \rangle = 1 \quad \text{if } n = e_1 + 3e_2 + \cdots + (2^r - 1)e_r.
\]

Considering the induced right \( \chi(A \otimes A_{(r)}Z_2)^* \)-module structure on \( \chi(A \otimes A_{(r)}Z_2)^* \) then yields the result that \( \chi Sq^n \) is nonzero in \( \chi(A \otimes A_{(r)}Z_2)^* \) if and only if there is a monomial of dimension \( n \) in \( \chi(A \otimes A_{(r)}Z_2)^* \) completing the proof of Lemma 1.

Proof of Lemma 2. By only a slight modification of the argument given by Peterson, in [3], to compute \( \chi(A \otimes A_{(r)}Z_2)^* \) and the fact that

\[
A^* = A^*/(\xi_1^{2^{r+1}}, \xi_2^{2^r}, ..., \xi_{r+1}^2, \xi_{r+2}, ...),
\]

we can show that as \( Z_2 \)-vector spaces

\[
\chi(A \otimes A_{(r)}Z_2)^* \cong \Lambda(\xi_1^2, \xi_2^{2^r - 1}, ..., \xi_r^2, \xi_{r+1})
\]

where the right-hand side is the exterior algebra over \( Z_2 \) generated by \( \xi_1^2, ..., \xi_{r+1}^2 \). We define a map \( \lambda: \Sigma^{2^{r+1}} \chi(A \otimes A_{(r)}Z_2) \rightarrow \chi(A \otimes A_{(r)}Z_2)^* \) of vector spaces over \( Z_2 \), by

\[
\lambda(\xi_1^{2^k} \xi_2^{2^k - 1} \cdots \xi_{r+1}^{2^k}) = \xi_1^{2^k(2^r - 1)} \xi_2^{2^k - 1} \cdots \xi_{r+1}^{2^k}.
\]
for $\epsilon_i = 0$ or 1. Since $\lambda$ is a monomorphism it follows that $A \otimes_{A_{-1}} \mathbb{Z}_2$ is nonzero in dimension $n$ if $\bigoplus_{j=0}^{(2^r+1)} \Sigma^j A_r \otimes_{A_{-1}} \mathbb{Z}_2$ is. Conversely we define a map

$$\rho: \chi(A \otimes_{A_{-1}} \mathbb{Z}_2)^* \to \bigoplus_{j=0}^{(2^r+1)} \Sigma^j \chi(A_r \otimes_{A_{-1}} \mathbb{Z}_2)^*$$

by the following procedure. Suppose that

$$\xi_1^{2^a_1} \xi_2^{2^a_2} \cdots \xi_{r+1}^{2^a_{r+1}} \cdots \xi_{s+r}^{2^a_{s+r}}, \quad j > r + 1, \ a_i \in \mathbb{Z},$$

is a monomial in $\chi(A \otimes_{A_{-1}} \mathbb{Z}_2)^*$ of dimension $n$. Begin by replacing it with the monomial

$$(3) \xi_1^{2^a_1+2^a_r \epsilon_1} \xi_2^{2^a_2} \cdots \xi_{r+1}^{2^a_{r+1}}$$

where $\omega = \sum_{i=r+1}^{s+r} a_i$ and $v = a_{r+2} + 3a_{r+3} + 7a_{r+4} + \cdots + (2^j - 1-1)a_j$. This monomial is also of dimension $n$. Next we inductively "reduce" the monomial while at the same time preserving its dimension. If (3) is of the form

$$\xi_1^{2^a_1} \xi_2^{2^a_2} \cdots \xi_{i-1}^{2^a_{i-1}} \xi_i^{2^a_i} \xi_i^{2^a_{i+1}} \cdots \xi_{s+r}^{2^a_{s+r}}$$

with $i < r + 1$, $b_i \in \mathbb{Z}$ and each $\epsilon_j$ either 0 or 1 we replace it with the monomial

$$(4) \xi_1^{2(b_i + 1) + 2^r + c + 2^{b_i-1}} \xi_2^{2^{b_i+1} + c + b_i + e} \cdots \xi_{s+r}^{2^{s+r} + e}$$

where $b_i = \epsilon_i + 2\epsilon_{i-1} + 4\epsilon_{i-2} + \cdots + 2^j - 1\epsilon_1 + 2 - 1$, each $\epsilon_j$ is either 0 or 1 and $c = \epsilon_{i-1} + 3\epsilon_{i-2} + \cdots + (2^j - 2 - 1)\epsilon_2 + (2^j - 1 - 1).$ The monomial (4) also has dimension $n$. Continuing in this fashion we end up with a monomial of the form

$$(5) \xi_1^{2^{s+r+1}} \xi_2^{2^{s+r+1}} \cdots \xi_{s+r}^{2^{s+r+1}}$$

with each $\epsilon_j$ zero or one. We define $\rho(\xi_1^{2^a_1} \xi_2^{2^a_2} \cdots \xi_{s+r}^{2^a_{s+r}})$ to be the nonzero monomial $\xi_1^{2^a_1} \cdots \xi_{s+r}^{2^a_{s+r}}$ in $\Sigma^{2^{s+r}} \chi(A_r \otimes_{A_{-1}} \mathbb{Z}_2)^*$ of dimension $n$; so

$$\bigoplus_{j=0}^{(2^r+1)} \Sigma^j \chi(A_r \otimes_{A_{-1}} \mathbb{Z}_2)^*$$

is nonzero in dimension $n$ completing the proof of Lemma 2.

**Added in proof.** Related results about the vanishing of Stiefel-Whitney classes have been proved by R. Stong. See §3 of *Cobordism and Stiefel-Whitney Numbers*, Topology 4 (1965), 241–246.

**References**

4. R. Stong, *Determination of $H^*(BO(k, \ldots, \infty))$ and $H^*(BU(k, \ldots, \infty))*, Trans. Amer. Math. Soc. 104 (1963), 526–544.

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