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TOTALLY REAL SUBMANIFOLDS IN A 6-SPHERE

NORIO EJIRI

ABSTRACT. A 6-dimensional sphere S^6 has an almost complex structure induced by properties of Cayley algebra. We investigate 3-dimensional totally real submanifolds in S^6 and classify 3-dimensional totally real submanifolds of constant sectional curvature.

1. Introduction. It is well known that a 6-dimensional (unit) sphere S^6 admits an almost Hermitian structure, which is a typical example of *Tachibana manifold* or a *nearly Kaehler manifold*.

There are two typical classes among all submanifolds of an almost Hermitian manifold: The one is the class of almost Hermitian Submanifolds and the other is the class of totally real submanifolds.

A. Gray [3] proved that S^6 *has no 4-dimensional almost Hermitian submanifolds*.

On the contrary, S^6 admits totally real submanifolds.

The purpose of this paper is to prove the following.

THEOREM 1. *A 3-dimensional totally real submanifold of S^6 is orientable and minimal.*

THEOREM 2. *Let M be a 3-dimensional totally real submanifold of constant curvature c in S^6. Then either $c = 1$ (i.e., M is totally geodesic) or $c = 1/16$.*

The latter case in Theorem 2 is locally equivalent to a minimal immersion $S^3(1/16) \to S^6$ defined by spherical harmonics of degree 6 [1].

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2. Almost Hermitian structures on S^6. Let $e_1, \ldots, e_7$ be the standard basis for R^7. Then the vector cross product in R^7 is defined by the table for $e_j \times e_k$.

TABLE

j/k	1	2	3	4	5	6	7
1	0	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
2	$-e_3$	0	e_1	e_6	$-e_7$	$-e_4$	e_5
3	e_2	$-e_1$	e	$-e_7$	$-e_6$	e_5	e_4
4	$-e_5$	$-e_6$	e_7	0	e_1	e_2	$-e_3$
5	e_4	e_7	e_6	$-e_1$	0	$-e_3$	$-e_2$
6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	0	e_1
7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	0

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We put $S^6 = \{x \in R^7;\ \|x\| = 1\}$ and define an almost complex structure J on S^6 by $JA = x \times A$, where $x \in S^6$ and $A \in T_x S^6$ (the tangent space of S^6 at x). It is easily seen that the Riemannian metric $\bar{g}$ on S^6 induced from R^7 is a Hermitian metric with respect to J. We denote by $\bar{\nabla}$ the covariant differentiation with respect to the Riemannian connection on S^6. Then we have the following (cf. for example [2]):

LEMMA 2.1. $(\bar{\nabla}_X J)X = 0$ *holds for all vector fields* X *on* S^6.

An almost Hermitian manifold with this property is called a *Tachibana manifold* or a *nearly Kaehler manifold.*

We define a skew-symmetric tensor field G of type (1, 2) by

$$G(X, Y) = \left(\bar{\nabla}_X J\right) Y.$$

Then we have

LEMMA 2.2. (i) $G(X, JY) = -JG(X, Y)$ *and*
(ii) $(\bar{\nabla}_X G)(Y, Z) = \bar{g}(Y, JZ)X + \bar{g}(X, Z)JY - \bar{g}(X, Y)JZ$
hold for all vector fields X, Y, Z *on* S^6.

3. 3-dimensional totally real submanifolds of S^6. Let (M, g) be a 3-dimensional totally real submanifold of $(S^6, J, \bar{g})$. We denote by ∇ the covariant differentiation on M. Then the second fundamental form σ of the immersion is given by

$$(X, Y) = \bar{\nabla}_X Y - \nabla_X Y \tag{3.1}$$

for vector fields X, Y on M. For a normal vector field ξ, we denote by $-A_\xi X$ and $\nabla^\perp_X \xi$ the tangential and normal components of $\bar{\nabla}_X \xi$ respectively so that

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi. \tag{3.2}$$

Then σ and A_ξ are related by $g(\sigma(X, Y), \xi) = g(A_\xi X, Y)$.

Let R and $R^\perp$ be the curvature tensor of ∇ and $\nabla^\perp$, respectively. Then the equations of Gauss, Codazzi and Ricci are given respectively by

$$\begin{aligned} g(R(X, Y)Z, W) = {} & g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \\ & + \bar{g}(\sigma(X, Z), \sigma(Y, W)) - \bar{g}(\sigma(X, W), \sigma(Y, Z)), \end{aligned} \tag{3.3}$$

$$(\nabla'_X \sigma)(Y, Z) - (\nabla'_Y \sigma)(X, Z) = 0, \tag{3.4}$$

$$g(R^\perp(X, Y)\xi, \eta) - g\left(\left[A_\xi, A_\eta\right]X, Y\right) = 0, \tag{3.5}$$

where $(\nabla'_X \sigma)(Y, Z) = \nabla^\perp_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$.

4. Proof of Theorem 1. Let (M, g) be a 3-dimensional totally real submanifold of $(S^6, J, \bar{g})$. First of all, we shall prove the following.

LEMMA 4.1. $G(X, Y)$ *is normal to* M *for* X, Y *tangent to* M.

PROOF. From (3.1) and (3.2) we have

$$g\left(\left(\bar{\nabla}_X J\right)Y, Z\right) = g(J\sigma(X, Z), Y) - g(J\sigma(X, Y), Z),$$

$$g\left(\left(\bar{\nabla}_Z J\right)X, Y\right) = g(J\sigma(Z, Y), X) - g(J\sigma(Z, X), Y),$$

$$g\left(\left(\bar{\nabla}_Y J\right)Z, X\right) = g(J\sigma(Y, X), Z) - g(J\sigma(Y, Z), X),$$

for X, Y, Z tangent to M. Since $\bar{g}$ is Hermitian with respect to J, $\bar{\nabla}_X J$ is skew-symmetric with respect to $\bar{g}$. This, together with Lemma 2.1, implies that the left-hand sides of the above three equations are equal to each other. Therefore we have $g((\bar{\nabla}_X J)Y, Z) = 0$, which means $G(X, Y)$ is orthogonal to M. Q.E.D.

By Lemma 2.2(i), we obtain

$$\begin{aligned}(\bar{\nabla}_X G)(JY, JZ) &= \bar{\nabla}_X G(JY, JZ) - G(\bar{\nabla}_X JY, JZ) - G(JY, \bar{\nabla}_X JZ) \\ &= -\bar{\nabla}_X G(Y, Z) - G((\bar{\nabla}_X J)Y, JZ) - G(J\bar{\nabla}_X Y, JZ) \\ &\quad - G(JY, (\bar{\nabla}_X J)Z) - G(JY, J\bar{\nabla}_X Z) \\ &\quad -\bar{\nabla}_X G(Y, Z) + JG(G(X, Y), Z) \\ &\quad + G(\bar{\nabla}_X Y, Z) + JG(Y, G(X, Z)) + G(Y, \bar{\nabla}_X Z) \\ &= -(\bar{\nabla}_X G)(Y, Z) + JG(G(X, Y), Z) + JG(Y, G(X, Z))\end{aligned}$$

for X, Y, Z tangent to M. This, combined with Lemma 2.2(ii), implies

$$G(Y, G(Z, X)) + G(Z, G(X, Y)) = g(X, Y)Z - g(X, Z)Y$$

and hence $G(X, G(Y, Z)) = g(X, Z)Y - g(X, Y)Z$ or equivalently

$$JG(X, JG(Y, Z)) = g(X, Z)Y - g(X, Y)Z \tag{4.1}$$

for X, Y, Z tangent to M. Since $JG(X, Y)$ is tangent to M by Lemma 4.1, we see from (4.1) that

$$g(JG(X, Y), Y)X - g(JG(X, Y), X)Y = JG(JG(X, Y), JG(X, Y)) = 0.$$

Thus $JG(X, Y)$ is orthogonal to X and Y if X and Y are linearly independent. This property, together with (4.1), implies that M is orientable, because the orientation can be defined by regarding $JG(X, Y)$ as the vector product of X and Y at each point of M.

Next, we shall prove that M is minimal. It follows immediately from (3.1), (3.2) and Lemma 4.1 that

$$\nabla^{\perp}_X JY = G(X, Y) + J\nabla_X Y \tag{4.2}$$

and

$$A_{JX} = -J\sigma(X, Y) \tag{4.3}$$

hold for X, Y tangent to M. By (3.1), (3.2), (4.2), (4.3) and Lemma 2.2(i), we obtain

$$\begin{aligned}(\bar{\nabla}_X G)(Y, Z) &= \bar{\nabla}_X G(Y, Z) - G(\bar{\nabla}_X Y, Z) - G(Y, \bar{\nabla}_X Z) \\ &= -A_{G(Y,Z)}X + \nabla^{\perp}_X G(Y, Z) - G(\bar{\nabla}_X Y, Z) - G(Y, \bar{\nabla}_X Z) \\ &= J\sigma(JG(Y, Z), X) + JG(X, G(Y, Z)) - J(\nabla_X JG)(Y, Z) \\ &\quad - G(\sigma(X, Y), Z) - G(Y, \sigma(X, Z))\end{aligned}$$

for X, Y, Z tangent to M. This, combined with Lemma 2.2(ii), implies

$$\begin{aligned}(\nabla_X JG)(Y, Z) = g(X, Y)Z - g(X, Z)Y + G(X, G(Y, Z)) + \sigma(X, JG(Y, Z)) \\ + JG(\sigma(X, Y), Z) + JG(Y, \sigma(Z, X)).\end{aligned}$$

Taking the normal component, we have

$$\sigma(X, JG(Y, Z)) + JG(\sigma(X, Y), Z) + JG(Y, \sigma(Z, X)) = 0 \tag{4.4}$$

for X, Y, Z tangent to M. Let e_1, e_2, e_3 be a local field of orthonormal frames on M. Then we may assume without loss of generality that $JG(e_1, e_2) = e_3$, $JG(e_2, e_3) = e_1$ and $JG(e_3, e_1) = e_2$. Hence we have from (4.4) that the trace of $\sigma = 0$, which implies that M is minimal.

5. Proof of Theorem 2. Let M be a 3-dimensional totally real submanifold of constant curvature c in S^6. Then the equation (3.3) of Gauss reduces to

$$(5.1)\quad \begin{aligned}(1-c)\{g(X,Z)g(Y,W)-g(X,W)g(Y,Z)\}\\ +\bar g(\sigma(X,Z),\sigma(Y,W))-\bar g(\sigma(X,W),\sigma(Y,Z))=0.\end{aligned}$$

If $c = 1$, then M is totally geodesic. Therefore it is sufficient to consider the case $c < 1$.

Consider a cubic function $f(X) = \bar g(\sigma(X, X), JX)$ defined on $\{X \in T_xM;\ \|X\| = 1\}$. If f attains its maximum at x, then $\bar g(\sigma(X, X), JY) = 0$ for Y orthogonal to X and hence $\sigma(X, X)$ is proportional to JX. Therefore, if f is constant, $\sigma(X, X) = 0$ for all X, since M is minimal. Thus f is not constant, since we are considering the case where M is not totally geodesic.

Choose e_1 to be the maximum point of f at each point $x \in M$. By the similar argument to the above, we see that f restricted to $\{X \in T_xM;\ \|X\| = 1$ and $g(X, e_1) = 0\}$ is not constant. Choose e_2 to be the maximum point of f restricted to $\{X \in T_xM;\ \|X\| = 1$ and $g(X, e_1) = 0\}$ and choose e_3 so that e_1, e_2, e_3 form an orthonormal frame field. Then we easily see that

$$(5.2)\qquad \bar g(\sigma(e_2, e_2), Je_3) = 0.$$

Put $a_i = \bar g(\sigma(e_i, e_i), Je_1)$. Then we have $a_1 + a_2 + a_3 = 0$, since M is minimal. We see that $a_1 > 0$, because a_1 is the maximum value for the cubic function f and M is not totally geodesic. Moreover, from (5.1) we have $1 - c + a_1a_2 - a_2^2 = 0$ and $1 - c + a_1a_3 - a_3^2 = 0$, since (4.3) implies that $\bar g(\sigma(X, Y), JZ)$ is symmetric in X, Y, Z. Therefore we get

$$(a_1, a_2, a_3) = \left(2\sqrt{(1-c)/3}\,, -\sqrt{(1-c)/3}\,, -\sqrt{(1-c)/3}\,\right),$$

which implies that

$$(5.3)\qquad \sigma(e_1, e_1) = 2\sqrt{(1-c)/3}\,Je_1$$

and

$$(5.4)\qquad \bar g(\sigma(X, X), Je_1) = -\sqrt{(1-c)/3}$$

for a unit vector X orthogonal to e_1. In particular, putting $X = (e_2 + e_3)/\sqrt{2}$, we obtain

$$(5.5)\qquad \bar g(\sigma(e_2, e_3), Je_1) = 0.$$

In consideration of (5.2), (5.3), (5.4), (5.5) and minimality of M, we may put $\sigma(e_2, e_2) = -\sqrt{(1-c)/3}\,Je_1 + \lambda Je_2$, $\sigma(e_3, e_3) = -\sqrt{(1-c)/3}\,Je_1 - \lambda e_2$, $\sigma(e_2, e_3) = -\lambda Je_3$. Putting $X = W = e_2$ and $Y = Z = e_3$ in (5.1), we obtain

$\lambda = \sqrt{2(1-c)/3}$. Therefore we have

$$(5.6)\qquad \begin{aligned} \sigma(e_2, e_2) &= -\sqrt{(1-c)/3}\, Je_1 + \sqrt{2(1-c)/3}\, Je_2, \\ \sigma(e_3, e_3) &= -\sqrt{(1-c)/3}\, Je_1 - \sqrt{2(1-c)/3}\, Je_2, \\ \sigma(e_2, e_3) &= -\sqrt{2(1-c)/3}\, Je_3, \end{aligned}$$

which, together with (5.3), (5.4) and (5.5), implies

$$(5.7)\qquad \sigma(e_1, e_2) = -\sqrt{(1-c)/3}\, Je_2, \qquad \sigma(e_1, e_3) = -\sqrt{(1-c)/3}\, Je_3.$$

Applying the equation (3.4) of Codazzi to (5.3), (5.6) and (5.7), we obtain $\nabla_{e_i} e_i = 0$, $\nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = -\frac{1}{4} e_3$, $\nabla_{e_1} e_3 = -\nabla_{e_3} e_1 = \frac{1}{4} e_2$, $\nabla_{e_2} e_3 = -\nabla_{e_3} e_2 = -\frac{1}{4} e_1$. Therefore we have $R(e_1, e_2)e_1 = 1/16 e_2$ and hence $c = 1/16$.

6. Remarks.

REMARK 1. Let M be a 3-dimensional totally real submanifold of S^6 and σ its second fundamental form. If we put $\tau = -J\sigma$, then τ is a symmetric tensor field of type (1, 2) on M and the equations of Gauss, Codazzi and Ricci can be written in terms of the intrinsic tensor field τ. By identifying the tangent bundle of M with the normal bundle, we can state the fundamental theorem in terms of intrinsic language of M. In particular, using a Killing frame e_1, e_3, e_3 on $S^3(1/16)$ (cf. for example [5]), we can give a minimal immersion of $S^3(1/16)$ into S^6 as a totally real submanifold.

REMARK 2. From Moore's theorem [4], we know that the minimum number l for which $S^3(c)$ can admit a (nontotally geodesic) minimal immersion into S^l is 6. This gives a counterexample for a problem in [**1**, p. 44].

REFERENCES

1. M. doCarmo and N. R. Wallach, *Minimal immersions of spheres into spheres*, Ann. of Math. (2) **95** (1971), 43–62.

2. A. Gray, *Minimal varieties and almost Hermitian submanifold*, Michigan Math. J. **12** (1965), 273–285.

3. ______, *Almost complex submanifolds in the six sphere*, Proc. Amer. Math. Soc. **20** (1969), 277–279.

4. J. D. Moore, *Isometric immersions of space forms into space forms*, Pacific J. Math. **40** (1972), 157–166.

5. S. Tanno, *Orthonormal frames on 3-dimensional Riemannian manifolds*, J. Differential Geom. **11** (1976), 467–474.

DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, TOKYO, JAPAN