TOTTALLY REAL SUBMANIFOLDS IN A 6-SPHERE

NORIO EJIRI

Abstract. A 6-dimensional sphere $S^6$ has an almost complex structure induced by properties of Cayley algebra. We investigate 3-dimensional totally real submanifolds in $S^6$ and classify 3-dimensional totally real submanifolds of constant sectional curvature.

1. Introduction. It is well known that a 6-dimensional (unit) sphere $S^6$ admits an almost Hermitian structure, which is a typical example of Tachibana manifold or a nearly Kaehler manifold.

There are two typical classes among all submanifolds of an almost Hermitian manifold: The one is the class of almost Hermitian Submanifolds and the other is the class of totally real submanifolds.

A. Gray [3] proved that $S^6$ has no 4-dimensional almost Hermitian submanifolds. On the contrary, $S^6$ admits totally real submanifolds.

The purpose of this paper is to prove the following.

Theorem 1. A 3-dimensional totally real submanifold of $S^6$ is orientable and minimal.

Theorem 2. Let $M$ be a 3-dimensional totally real submanifold of constant curvature $c$ in $S^6$. Then either $c = 1$ (i.e., $M$ is totally geodesic) or $c = 1/16$.

The latter case in Theorem 2 is locally equivalent to a minimal immersion $S^3(1/16) \to S^6$ defined by spherical harmonics of degree 6 [1].

The author is grateful to Professor K. Ogiue for his useful criticism.

2. Almost Hermitian structures on $S^6$. Let $e_1, \ldots, e_7$ be the standard basis for $R^7$. Then the vector cross product in $R^7$ is defined by the table for $e_j \times e_k$.

<table>
<thead>
<tr>
<th>$j/k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$e_3$</td>
<td>$-e_2$</td>
<td>$e_5$</td>
<td>$-e_4$</td>
<td>$e_7$</td>
<td>$-e_6$</td>
</tr>
<tr>
<td>2</td>
<td>$-e_3$</td>
<td>0</td>
<td>$e_1$</td>
<td>$e_6$</td>
<td>$-e_7$</td>
<td>$-e_4$</td>
<td>$e_5$</td>
</tr>
<tr>
<td>3</td>
<td>$e_2$</td>
<td>$-e_1$</td>
<td>$e$</td>
<td>$-e_7$</td>
<td>$-e_6$</td>
<td>$e_5$</td>
<td>$e_4$</td>
</tr>
<tr>
<td>4</td>
<td>$-e_5$</td>
<td>$-e_6$</td>
<td>$e_7$</td>
<td>0</td>
<td>$e_1$</td>
<td>$e_2$</td>
<td>$-e_3$</td>
</tr>
<tr>
<td>5</td>
<td>$e_4$</td>
<td>$e_7$</td>
<td>$e_6$</td>
<td>$-e_1$</td>
<td>0</td>
<td>$-e_3$</td>
<td>$-e_2$</td>
</tr>
<tr>
<td>6</td>
<td>$-e_7$</td>
<td>$e_4$</td>
<td>$-e_5$</td>
<td>$-e_2$</td>
<td>$e_3$</td>
<td>0</td>
<td>$e_1$</td>
</tr>
<tr>
<td>7</td>
<td>$e_6$</td>
<td>$-e_5$</td>
<td>$-e_4$</td>
<td>$e_3$</td>
<td>$e_2$</td>
<td>$-e_1$</td>
<td>0</td>
</tr>
</tbody>
</table>

Received by the editors January 26, 1981.

1980 Mathematics Subject Classification. Primary 53C40, 53C42; Secondary 58D10.

Key words and phrases. Cayley algebra, Tachibana space, nearly Kaehler manifold, totally real submanifold, Killing frame.

© 1981 American Mathematical Society

0002-9939/81/0000-0569/$02.25

759
We put $S^6 = \{ x \in R^7; \| x \| = 1 \}$ and define an almost complex structure $J$ on $S^6$ by $J A = x \times A$, where $x \in S^6$ and $A \in T_x S^6$ (the tangent space of $S^6$ at $x$). It is easily seen that the Riemannian metric $\bar{g}$ on $S^6$ induced from $R^7$ is a Hermitian metric with respect to $J$. We denote by $\bar{\nabla}$ the covariant differentiation with respect to the Riemannian connection on $S^6$. Then we have the following (cf. for example [2]):

**Lemma 2.1.** $(\bar{\nabla}_x J) X = 0$ holds for all vector fields $X$ on $S^6$.

An almost Hermitian manifold with this property is called a Tachibana manifold or a nearly Kaehler manifold.

We define a skew-symmetric tensor field $G$ of type (1, 2) by

$$G(X, Y) = (\bar{\nabla}_x J) Y.$$

Then we have

**Lemma 2.2.** (i) $G(X, JY) = -JG(X, Y)$ and
(ii) $(\bar{\nabla}_x G)(Y, Z) = \bar{g}(\bar{g}(Y, JZ)X + \bar{g}(X, Z)JY - \bar{g}(X, Y)JZ$ hold for all vector fields $X, Y, Z$ on $S^6$.

3. 3-dimensional totally real submanifolds of $S^6$. Let $(M, g)$ be a 3-dimensional totally real submanifold of $(S^6, J, \bar{g})$. We denote by $\nabla$ the covariant differentiation on $M$. Then the second fundamental form $\sigma$ of the immersion is given by

$$(3.1) \quad (X, Y) = \bar{\nabla}_X Y - \nabla_X Y$$

for vector fields $X, Y$ on $M$. For a normal vector field $\xi$, we denote by $-A_\xi X$ and $\nabla_X^\perp \xi$ the tangential and normal components of $\nabla_X \xi$ respectively so that

$$(3.2) \quad \nabla_X \xi = -A_\xi X + \nabla_X^\perp \xi.$$

Then $\sigma$ and $A_\xi$ are related by $g(\sigma(X, Y), \xi) = g(A_\xi X, Y)$.

Let $R$ and $R^\perp$ be the curvature tensor of $\nabla$ and $\nabla^\perp$, respectively. Then the equations of Gauss, Codazzi and Ricci are given respectively by

$$(3.3) \quad g(R(X, Y)Z, W) = g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + \bar{g}(\sigma(X, Z), \sigma(Y, W)) - \bar{g}(\sigma(X, W), \sigma(Y, Z)),$$

$$(3.4) \quad (\nabla_X \sigma)(Y, Z) - (\nabla_Y \sigma)(X, Z) = 0,$$

$$g(R^\perp(X, Y)\xi, \eta) - g([A_\xi, A_\eta] X, Y) = 0,$$

where $(\nabla_X \sigma)(Y, Z) = \nabla_X^\perp \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$.

4. Proof of Theorem 1. Let $(M, g)$ be a 3-dimensional totally real submanifold of $(S^6, J, \bar{g})$. First of all, we shall prove the following.

**Lemma 4.1.** $G(X, Y)$ is normal to $M$ for $X, Y$ tangent to $M$.

**Proof.** From (3.1) and (3.2) we have

$$g((\bar{\nabla}_x J) Y, Z) = g(J\sigma(X, Z), Y) - g(J\sigma(X, Y), Z),$$

$$g((\bar{\nabla}_x J) Y, Z) = g(J\sigma(Z, Y), X) - g(J\sigma(Z, X), Y),$$

$$g((\bar{\nabla}_x J) Y, Z) = g(J\sigma(Y, X), Z) - g(J\sigma(Y, Z), X),$$

$$g((\bar{\nabla}_x J) Y, Z) = g(J\sigma(Y, Z), X) - g(J\sigma(Y, X), Z),$$

$$g((\bar{\nabla}_x J) Y, Z) = g(J\sigma(Z, X), Y) - g(J\sigma(Z, Y), X).$$
for $X, Y, Z$ tangent to $M$. Since $\tilde{g}$ is Hermitian with respect to $J$, $\tilde{\nabla}_x J$ is skew-symmetric with respect to $\tilde{g}$. This, together with Lemma 2.1, implies that the left-hand sides of the above three equations are equal to each other. Therefore we have $g((\tilde{\nabla}_x J) Y, Z) = 0$, which means $G(X, Y)$ is orthogonal to $M$. Q.E.D.

By Lemma 2.2(i), we obtain

$$(\tilde{\nabla}_x G)(JY, JZ) = \tilde{\nabla}_x G(JY, JZ) - G(\tilde{\nabla}_x JY, JZ) - G(JY, \tilde{\nabla}_x JZ)$$

$$= -\tilde{\nabla}_x G(Y, Z) - G((\tilde{\nabla}_x J) Y, JZ) - G(JY, J\tilde{\nabla}_x Z)$$

$$- G(JY, (\tilde{\nabla}_x J) Z) - G(JY, J\tilde{\nabla}_x Z)$$

$$+ G(\tilde{\nabla}_x Y, Z) + JG(G(X, Y), Z)$$

$$+ G(Y, \tilde{\nabla}_x Z)$$

$$= - (\tilde{\nabla}_x G)(Y, Z) + JG(G(X, Y), Z) + JG(G(Y, X), Z) + JG(Y, G(X, Z))$$

for $X, Y, Z$ tangent to $M$. This, combined with Lemma 2.2(ii), implies

$$G(Y, G(Z, X)) + G(Z, G(X, Y)) = g(X, Y) Z - g(X, Z) Y$$

and hence $G(X, G(Y, Z)) = g(X, Z) Y - g(X, Y) Z$ or equivalently

$$(4.1) \quad JG(X, JG(Y, Z)) = g(X, Z) Y - g(X, Y) Z$$

for $X, Y, Z$ tangent to $M$. Since $JG(X, Y)$ is tangent to $M$ by Lemma 4.1, we see from (4.1) that

$$g(JG(X, Y), Y) X - g(JG(X, Y), X) Y = JG(JG(X, Y), JG(X, Y)) = 0.$$
for $X$, $Y$, $Z$ tangent to $M$. Let $e_1$, $e_2$, $e_3$ be a local field of orthonormal frames on $M$. Then we may assume without loss of generality that $JG(e_1, e_2) = e_3$, $JG(e_2, e_3) = e_1$ and $JG(e_3, e_1) = e_2$. Hence we have from (4.4) that the trace of $\sigma = 0$, which implies that $M$ is minimal.

5. Proof of Theorem 2. Let $M$ be a 3-dimensional totally real submanifold of constant curvature $c$ in $S^6$. Then the equation (3.3) of Gauss reduces to

\[(1 - c)\left( g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \right) + \bar{g}(\sigma(X, Z), \sigma(Y, W)) - \bar{g}(\sigma(X, W), \sigma(Y, Z)) = 0.\]

If $c = 1$, then $M$ is totally geodesic. Therefore it is sufficient to consider the case $c < 1$.

Consider a cubic function $f(X) = \bar{g}(\sigma(X, X), JX)$ defined on \{ $X \in T_x M; \|X\| = 1$ \}. If $f$ attains its maximum at $x$, then $\bar{g}(\sigma(X, X), JY) = 0$ for $Y$ orthogonal to $X$ and hence $\sigma(X, X)$ is proportional to $JX$. Therefore, if $f$ is constant, $\sigma(X, X) = 0$ for all $X$, since $M$ is minimal. Thus $f$ is not constant, since we are considering the case where $M$ is not totally geodesic.

Choose $e_1$ to be the maximum point of $f$ at each point $x \in M$. By the similar argument to the above, we see that $f$ restricted to \{ $X \in T_x M; \|X\| = 1$ and $g(X, e_1) = 0$ \} is not constant. Choose $e_2$ to be the maximum point of $f$ restricted to \{ $X \in T_x M; \|X\| = 1$ and $g(X, e_1) = 0$ \} and choose $e_3$ so that $e_1$, $e_2$, $e_3$ form an orthonormal frame field. Then we easily see that

\[(5.2) \quad \bar{g}(\sigma(e_2, e_2), Je_3) = 0.\]

Put $a_i = \bar{g}(\sigma(e_i, e_i), Je_i)$. Then we have $a_1 + a_2 + a_3 = 0$, since $M$ is minimal. We see that $a_1 > 0$, because $a_1$ is the maximum value for the cubic function $f$ and $M$ is not totally geodesic. Moreover, from (5.1) we have $1 - c + a_1a_2 - a_2^2 = 0$ and $1 - c + a_1a_3 - a_3^2 = 0$, since (4.3) implies that $\bar{g}(\sigma(X, Y), JZ)$ is symmetric in $X$, $Y$, $Z$. Therefore we get

\[(a_1, a_2, a_3) = \left( 2\sqrt{(1 - c)/3}, -\sqrt{(1 - c)/3}, -\sqrt{(1 - c)/3} \right),\]

which implies that

\[(5.3) \quad \sigma(e_1, e_1) = 2\sqrt{(1 - c)/3} \ Je_1\]

and

\[(5.4) \quad \bar{g}(\sigma(X, X), Je_1) = -\sqrt{(1 - c)/3}\]

for a unit vector $X$ orthogonal to $e_1$. In particular, putting $X = (e_2 + e_3)/\sqrt{2}$, we obtain

\[(5.5) \quad \bar{g}(\sigma(e_2, e_3), Je_1) = 0.\]

In consideration of (5.2), (5.3), (5.4), (5.5) and minimality of $M$, we may put $\sigma(e_2, e_2) = -\sqrt{(1 - c)/3} \ Je_1 + \lambda Je_2$, $\sigma(e_3, e_3) = -\sqrt{(1 - c)/3} \ Je_1 - \lambda e_2$, $\sigma(e_2, e_3) = -\lambda Je_3$. Putting $X = W = e_2$ and $Y = Z = e_3$ in (5.1), we obtain
\[ \lambda = \sqrt{2(1 - c)/3}. \] Therefore we have

\[ \sigma(e_2, e_2) = -\sqrt{(1 - c)/3} J e_1 + \sqrt{2(1 - c)/3} J e_2, \]

\[ \sigma(e_3, e_3) = -\sqrt{(1 - c)/3} J e_1 - \sqrt{2(1 - c)/3} J e_2, \]

\[ \sigma(e_2, e_3) = -\sqrt{2(1 - c)/3} J e_3, \]

which, together with (5.3), (5.4) and (5.5), implies

\[ \sigma(e_1, e_2) = -\sqrt{(1 - c)/3} J e_2, \quad \sigma(e_1, e_3) = -\sqrt{(1 - c)/3} J e_3. \]

Applying the equation (3.4) of Codazzi to (5.3), (5.6) and (5.7), we obtain \( \nabla e_i e_j = 0 \), \( \nabla e_1 e_2 = -\frac{1}{4} e_3 \), \( \nabla e_1 e_3 = -\frac{1}{4} e_2 \), \( \nabla e_2 e_3 = -\frac{1}{4} e_1 \). Therefore we have \( R(e_1, e_2) e_1 = 1/16 e_2 \) and hence \( c = 1/16 \).


Remark 1. Let \( M \) be a 3-dimensional totally real submanifold of \( S^6 \) and \( \sigma \) its second fundamental form. If we put \( \tau = -J \sigma \), then \( \tau \) is a symmetric tensor field of type \((1, 2)\) on \( M \) and the equations of Gauss, Codazzi and Ricci can be written in terms of the intrinsic tensor field \( \tau \). By identifying the tangent bundle of \( M \) with the normal bundle, we can state the fundamental theorem in terms of intrinsic language of \( M \). In particular, using a Killing frame \( e_1, e_3, e_3 \) on \( S^3(1/16) \) (cf. for example [5]), we can give a minimal immersion of \( S^3(1/16) \) into \( S^6 \) as a totally real submanifold.

Remark 2. From Moore's theorem [4], we know that the minimum number \( l \) for which \( S^3(c) \) can admit a (nontotally geodesic) minimal immersion into \( S^l \) is 6. This gives a counterexample for a problem in [1, p. 44].

References


Department of Mathematics, Tokyo Metropolitan University, Tokyo, Japan