AN INVARIANT FOR CONTINUOUS FACTORS OF MARKOVhifts

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Abstract. Let $\Sigma_A$ and $\Sigma_B$ be subshifts of finite type with Markov measures $(\rho, P)$ and $(q, Q)$. It is shown that if there is a continuous onto measure-preserving factor map from $\Sigma_A$ to $\Sigma_B$, then the block of the Jordan form of $Q$ with nonzero eigenvalues is a principal submatrix of the Jordan form of $P$. If $\Sigma_A$ and $\Sigma_B$ are irreducible with the same topological entropy, then the same relationship holds for the matrices $A$ and $B$. As a consequence, $\zeta_B(t)/\zeta_A(t)$, the ratio of the zeta functions, is a polynomial. From this it is possible to construct a pair of equal-entropy subshifts of finite type that have no common equal-entropy continuous factor of finite type, and a strictly sofic system that cannot have an equal-entropy subshift of finite type as a continuous factor.

1. Introduction. Let $A$ be an $l \times l$ matrix of 0's and 1's. The subshift of finite type, $\Sigma_A$, determined by $A$ is the closed invariant subspace of $\{1, \ldots, l\}^\mathbb{Z}$ consisting of all $x = \ldots x_{-1}x_0x_1\ldots$ such that $A^i x_{k+i} = 1$ for all $i$, together with the shift transformation $T_A$. A Markov measure is defined on this space by a pair $(\rho, P)$, where $P$ is a stochastic matrix compatible with $A$ (i.e., positive, row-sum 1, and $P_{ij} > 0$ exactly when $A_{ij} = 1$) and $\rho$ is a probability vector with $\rho P = \rho$.

The following dynamical properties of subshifts of finite type will be used. A subshift of finite type is topologically transitive when its transition matrix is irreducible [9]. The irreducibility of the transition matrix is also the condition needed for ergodicity with respect to any Markov measure [9]. The zeta function of a dynamical system is $\zeta(t) = \exp[\sum_{n=1}^{\infty}(N_n) t^n/n]$, where $N_n$ is the number of points fixed by the $n$th power of the transformation. For a subshift of finite type, $\Sigma_A$, this has the form $\zeta_A(t) = [t^l C_A(1/t)]^{-1}$, where $C_A(x)$ is the characteristic polynomial of $A$ [2]. The topological entropy of an irreducible subshift of finite type is $\log \lambda$, where $\lambda$ is the largest positive real eigenvalue [9], and the measure-theoretic entropy with respect to a Markov measure $(\rho, P)$ is $-\sum_{ij} \rho_i P_{ij} \log P_{ij}$ [9]. An irreducible subshift of finite type has a unique measure of maximal entropy [9], which is a Markov measure whose matrix has the form $P = \frac{1}{\lambda} R^{-1} A R$ (R a diagonal matrix with strictly positive diagonal entries). Any continuous onto finite-to-one factor map between two subshifts of finite type will carry the measure of maximal entropy of one to the measure of maximal entropy of the other [3]. The Curtis-Hedlund-Lyndon theorem [6] asserts that any continuous factor map between subshifts of finite type is a block map composed with some power of the shift.

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825
means that by going to a higher block presentation of the domain shift, any continuous factor map can be expressed as a one-block map [1].

2. P’s and Q’s.

**Theorem.** If $\Sigma_A$ and $\Sigma_B$ are subshifts of finite type and if there is a continuous onto factor map between them that takes $(p, P)$ to $(q, Q)$, then the block of the Jordan form of $Q$ with nonzero eigenvalues is a principal submatrix of the Jordan form of $P$.

**Proof.** We may assume that the factor map $\phi: \Sigma_A \to \Sigma_B$ is a one-block map [10]. Take $L_A$ to be the alphabet of $\Sigma_A$ and $L_B$ the alphabet of $\Sigma_B$. $R$ will be the relation matrix that represents the equivalence relation defined on $L_A$ by $\phi$; for $i \in L_A$, $a \in L_B$

$$R_{ia} = \begin{cases} 1 & \text{if } \phi(i) = a, \\ 0 & \text{otherwise.} \end{cases}$$

A time-zero cylinder set is defined by $[a_1, \ldots, a_n] = \{x: x_0 = a_1, \ldots, x_{n-1} = a_n\}$. The inverse image of a time-zero cylinder set in $\Sigma_B$ is a finite union of time-zero cylinder sets in $\Sigma_A$, of the same length. For any $[a_1, \ldots, a_n] \subseteq \Sigma_B$ define $u^{[a_1, \ldots, a_n]} \in \mathbb{R}^{|L_A|}$ and $v^{[a_1, \ldots, a_n]} \in \mathbb{R}^{|L_B|}$ by $(u^{[a_1, \ldots, a_n]})_i = \sum p_{i,i'} P_{i'} \cdots P_{i_{n-1}}$, where the sum is taken over all $[i_1, \ldots, i_n]$ in $\phi^{-1}[a_1, \ldots, a_n]$ such that $i_n = i$,

$$(v^{[a_1, \ldots, a_n]})_a = \begin{cases} q_{a,a_1} Q_{a,a_2} \cdots Q_{a_{n-1},a} & \text{if } a_n = a, \\ 0 & \text{otherwise.} \end{cases}$$

Let $U$ be the collection of all $u^{[a_1, \ldots, a_n]}$ and $V$ that of all $v^{[a_1, \ldots, a_n]}$. $V$ generates all of $\mathbb{R}^{|L_A|}$; let $\mathbb{Q}_U$ be the subspace of $\mathbb{R}^{|L_A|}$ generated by $U$. A computation using the measure-preserving property of $\phi$ shows the diagram

$$\begin{array}{ccc} \mathbb{Q}_U & \xrightarrow{P} & \mathbb{Q}_U \\ \downarrow R & & \downarrow R \\ \mathbb{R}^{|L_A|} & \xrightarrow{Q} & \mathbb{R}^{|L_B|} \end{array}$$

commutes, where the matrices operate by left multiplication, i.e. $xPR = xRQ$ for all $x \in \mathbb{Q}_U$. Since $R$ has rank $|L_B|$, we have the desired result. Notice that this linear algebra situation is equivalent to the existence of such a factor map.

**Corollary A.** If $\Sigma_A$, $\Sigma_B$ are irreducible subshifts of finite type and $\Sigma_B$ is a continuous finite-to-one factor of $\Sigma_A$, then the block of the Jordan form of $B$ with nonzero eigenvalues is a principal submatrix of the Jordan form of $A$. In particular, if $\xi_A(t)$ and $\xi_B(t)$ are the zeta functions, then $\xi_B(t)/\xi_A(t)$ is a polynomial.

**Proof.** $\Sigma_A$ and $\Sigma_B$ have the same topological entropy, $\log \lambda$. The matrices $P$, $Q$ for the measures of maximal entropy are $P = \lambda R^{-1}AR$ and $Q = \lambda S^{-1}BS$, where $R$, $S$ are the appropriate diagonal matrices of full rank. The desired result is obtained by applying the theorem. Recalling that $\xi_A(t) = (t^{1\cdot\lambda}|C_A(1/t))^{-1}$, we have the observation about the ratio of the zeta functions. We also have a completely topological proof of this corollary, and M. Nasu [8] has proved the fact about the ratio of the zeta functions using graph-theoretic techniques.
Corollary B. There exist equal-entropy mixing [1] subshifts of finite type that have no common equal-entropy continuous factor. This is in contrast to the Adler-Marcus Theorem, which asserts that any such shifts have a common equal-entropy continuous extension [1].

Proof. Take

\[ A = \begin{bmatrix} 001 \\ 101 \\ 010 \end{bmatrix}, \quad B = \begin{bmatrix} 00001 \\ 10000 \\ 01000 \\ 00100 \\ 00011 \end{bmatrix}, \]

then \( C_A(x) = x^3 - x - 1 \), which is irreducible, and \( C_B(x) = x^5 - x^4 - 1 = (x^3 - x - 1)(x^2 - x + 1) \). These are both mixing and have the same entropy. Since \( C_A(x) \) is irreducible over \( \mathbb{Z} \), any continuous finite-to-one factor of \( \Sigma_A \) must have the same zeta function. \( \Sigma_B \) has a fixed point, so any factor of it must have a fixed point. There is no subshift of finite type that meets both of these requirements.

Corollary C. There exists a mixing strictly sofic system [11] that has no equal-entropy subshift of finite type as a continuous factor. This should be compared to the fact that any sofic system is a continuous equal-entropy factor of a subshift of finite type [4].

Proof. Begin with \( \Sigma_A \) as in the previous corollary. Obtain a strictly sofic system of the same entropy by identifying the pair of two-blocks [2, 3] and [3, 2]. This sofic system has a fixed point. Any subshift of finite type that is an equal-entropy continuous factor of this system is also one for \( \Sigma_A \). We already know there is no such shift. This construction was noticed by Brian Marcus.

Corollary D. Any equal-entropy continuous factor of the full shift which is a subshift of finite type is shift equivalent (in the sense of Williams [12]) to the same full shift. This was previously proved in [7].

Proof. Any equal-entropy continuous factor of the full \( n \)-shift that is a subshift of finite type has the zeta function \((1 - nt)^{-1}\). R. Williams [12] has shown that any subshift of finite type with this zeta function is shift equivalent to the full \( n \)-shift. It is possible to deduce Corollaries B and C from the work of J. Cuntz and W. Krieger [5].

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