INVOLUTIONS WITH FIXED POINT SET
OF CONSTANT CODIMENSION

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ABSTRACT. The cobordism classes of manifolds admitting involutions with fixed point set of codimension 5 are determined by means of Stiefel-Whitney classes.

1. Introduction. Let \( \mathcal{R}_n \) be the group of nonoriented cobordism classes of \( n \)-dimensional smooth manifolds and let \( J_n^k \) be its subset consisting of the classes which are represented by manifolds admitting smooth involutions with fixed point set of constant codimension \( k \). \( J_n^k \) is a subgroup of \( \mathcal{R}_n \) and \( J_\ast^k = \sum_{n-k}^\infty J_n^k \) is an ideal of the nonoriented cobordism ring \( \mathcal{R}_\ast = \sum_{n=0}^\infty \mathcal{R}_n \). Capobianco [1] proved the following results:

**Proposition 1.** \( J_n^3 \) is the set of classes \( \alpha \) in \( \mathcal{R}_n \) with \( W_{i}W_{j-1}(\alpha) = W_{i+2}W_{j-3}(\alpha) = 0 \), for each \( i, j, 0 < j < n, 5 < i < n \).

**Proposition 2.** \( J_n^3 \subseteq J_\ast^3 \).

In this note, we shall prove

**Theorem.** \( J_n^5 \) is the set of classes \( \alpha \) in \( J_\ast^3 \) with \( W_{n-2}W_2^4(\alpha) = W_{n-9}W_2^3W_3(\alpha) = W_{n-10}W_2^2W_3^2(\alpha) = W_{n-11}W_2W_3^3(\alpha) = W_{n-12}W_3^4(\alpha) = 0 \).

2. Characteristic numbers of classes in \( J_n^5 \). Let \( \xi \to V \) be a smooth \( k \)-plane bundle over a closed smooth manifold \( V \) and let \( \pi: RP(\xi) \to V \) be the associated projective space bundle. Denote by \( a \) the characteristic class of the canonical line bundle \( \lambda \to RP(\xi) \). Then by [2, \$21], \( H^*(RP(\xi); \mathbb{Z}_2) \) is the free \( H^*(V; \mathbb{Z}_2) \)-module via \( \pi^* \) on the classes \( 1, a, \ldots, a^{k-1} \), subject to the relation \( \sum_{j=0}^{k} a^{k-j}\pi^*(v_j) = 0 \), where \( v_j \) is the \( j \)th Whitney class of \( \xi \). The total Stiefel-Whitney class of \( RP(\xi) \) is given by

\[
W(RP(\xi)) = \pi^*(W(V))\left(\sum_{j=0}^{k} (1 + a)^{k-j}\pi^*(v_j)\right).
\]

Now suppose that a class \( \alpha \) is represented by a manifold \( M^n \) admitting an involution with fixed point set \( F \) of codimension \( k \). Let \( q: \nu \to F \) be the normal bundle. Then by [2, (22.2)], \( \alpha \) is the class of \( RP(\nu \oplus R) \), which is the total space of the projective space bundle associated to \( p: \nu \oplus R \to F \). Let \( e \), resp. \( c \), be the characteristic class of the canonical line bundle \( \lambda \to RP(\nu \oplus R) \), resp. \( \lambda \to RP(\nu) \).
Then we have

**Proposition 3.** For any \( x \in H^j(F; Z_2), 0 < j < n - k, \)

\[
\langle p^*(x)e^{n-j}, [RP(\nu \oplus R)] \rangle = \langle q^*(x)e^{n-1-j}, [RP(\nu)] \rangle.
\]

The proof can be found in [5]. It follows from [2, §25]

**Proposition 4.** If \( q^*(x)c^{n-1} \) represents a characteristic class of \( \lambda \to RP(\nu), \) then

\[
\langle p^*(x)e^{n-j}, [RP(\nu \oplus R)] \rangle = 0.
\]

Let us apply these facts to the case of constant codimension 5.

**Lemma 5.** If \( \alpha \in J'_*, \) then \( W^{n-8}W_2^2(a) = W^{n-9}W_2^3W_3(\alpha) = W^{n-10}W_2^2W_3^2(\alpha) = W^{n-11}W_2^3W_3^2(\alpha) = 0. \)

**Proof.** Denote \( W'_j = W'_j(RP(\nu \oplus R)), W'_j = W'_j(RP(\nu)) \) and let \( v_j, \) resp. \( w_j, \) be the \( j \)th Whitney class of \( \nu, \) resp. \( F. \) By Propositions 3 and 4,

\[
\begin{align*}
W^{n-8}W_2^4 & = p^*\{(w_1 + v_1)^{n-8}w_1\} e^8, \\
(W'_1 + c)^{n-9}(W'_2 + W'_1c)(W'_3 + W'_2c)c^3 + (W'_1 + c)(W'_2 + W'_1c)^2c^3 \\
& \quad + (W'_1 + c)c^7 + (W'_2 + W'_1c + c^2)^4 \\
& = q^*\{(w_1 + v_1)^{n-8}w_1\} c^7,
\end{align*}
\]

\[
\begin{align*}
W^{n-10}W_2^2W_3^2 & = p^*\{(w_1 + v_1)^{n-10}w_2\} e^8, \\
(W'_1 + c)^{n-10}\{W'_2W'_2c^3 + W'_3^2c^3 + W'_2^2c\} & = q^*\{(w_1 + v_1)^{n-10}w_1\} c^7,
\end{align*}
\]

\[
\begin{align*}
W^{n-11}W_2^3W_3^2 & = p^*\{(w_1 + v_1)^{n-11}w_3\} e^8, \\
(W'_1 + c)^{n-11}\{W'_2W'_2W'_3^2 + W'_3W'_2W'_2c + (W'_2^4 + W'_1W'_2W'_3^2 + W'_1W'_2W'_3^2) \}
& \quad + (W'_1W'_2^3 + W'_1W'_3)c^3 \\
& = q^*\{(w_1 + v_1)^{n-11}w_3\} c^7,
\end{align*}
\]

\[
\begin{align*}
W^{n-12}W_4^4 & = p^*\{(w_1 + v_1)^{n-12}w_4\} e^8, \\
(W'_1 + c)^{n-12}W_2^2c^3 & = q^*\{(w_1 + v_1)^{n-12}w_4\} c^7.
\end{align*}
\]

3. **A system of generators of \( J'_*. \)** As is well known, \( R_* \) is a graded polynomial algebra over \( Z_2 \) with one generator in each dimension \( n \) which is not of the form \( 2^r - 1. \) We shall choose a suitable system of generators of \( R_* \) for our purpose. Let \( (n_1, n_2, \ldots, n_{2k}) \) be a \( 2k \)-tuple of nonnegative integers with \( n_1 + n_2 + \cdots + n_{2k} = n - 2k + 1. \) We denote by \( RP(n_1, n_2, \ldots, n_{2k}) \) the projective space bundle associated to the bundle \( \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_{2k} \) over \( RP(n_1) \times RP(n_2) \times \cdots \times RP(n_{2k}), \) where \( \lambda_i (i = 1, 2, \ldots, 2k) \) is the pull-back of the canonical line bundle over the \( i \)th factor. Stong [4, Lemma 3.4] proved that
RP(n_1, n_2, \ldots, n_{2k}) belongs to \mathcal{J}^k_m and is indecomposable in \mathcal{M}_* if and only if
\[
\left(\begin{array}{c}
  n_1
\end{array}\right) + \left(\begin{array}{c}
  n_2
\end{array}\right) + \cdots + \left(\begin{array}{c}
  n_{2k}
\end{array}\right)
\]
is odd. First, we shall show

**Lemma 6.** For each \( n > 13 \), not of the form \( 2^r - 1 \) or \( 2^r - 2 \), there exists a generator \( u_n \in \mathcal{J}^5_m \) which is indecomposable in \( \mathcal{M}_* \).

**Proof.** If \( \left(\begin{array}{c}
  n\
  2
\end{array}\right) \equiv 0 \mod 2 \), \( RP(n - 9, 0, \ldots, 0) \) (9 zeroes) is indecomposable in \( \mathcal{M}_* \). Consider the case \( \left(\begin{array}{c}
  n\
  2
\end{array}\right) \equiv 1 \mod 2 \). Let \( n - 1 = 2^r + 2^{r+1} + \cdots + 2^{r+k}, \ r_1 > r_2 > \cdots > r_k > 0 \). Since \( \left(\begin{array}{c}
  n\
  2
\end{array}\right) = \left(\begin{array}{c}
  n - 1\
  8
\end{array}\right) \), \( \{r_1, r_2, \ldots, r_k\} \) contains 3. When \( \{r_1, r_2, \ldots, r_k\} \) does not contain 1, \( RP(n - 11, 1, 1, 0, \ldots, 0) \) (7 zeroes) is indecomposable in \( \mathcal{M}_* \). Finally, suppose that \( \{r_1, r_2, \ldots, r_k\} \) contains both 1 and 2. Since \( n - 1 \) is not of the form \( 2^r - 1 \) or \( 2^r - 2 \), there exists a number \( i \) such that \( r_i > r_{i+1} + 1 \). Then, \( RP(2r_1 + \cdots + 2r_k - 2, 2r_1 + \cdots + 2r_k - 14, 8, 0, \ldots, 0) \) (7 zeroes) is indecomposable in \( \mathcal{M}_* \).

Let \( x_2 \) be the class of \( RP(2) \) and let \( x_{2^n} \) be the class of \( RP(2^n) \cup RP(2^n) \) for \( n > 1 \). Denote by \( y_n \) (\( n = 5, 6 \)) the class of \( RP(n - 3, 3, 0, 0, 0, 0) \) and by \( z_n \) (\( n = 9, 10, 12 \)) the class of \( RP(n - 5, 0, 0, 0, 0, 0, 0, 0) \). Furthermore, by \( [3, \S 7, \text{Remark}] \) we know that there exists a class \( z_{11} \) of an indecomposable manifold which belongs to \( J^5_{11} \). Thus we have

**Lemma 7.** \( \mathcal{M}_* \) is a polynomial algebra over \( \mathbb{Z}_2 \) with the system of generators: \( \{x_2, x_{2^n} (n = 1, 2, \ldots), u_n (n > 13, n \neq 2^r, 2^r - 1), y_5, y_6, z_9, z_{10}, z_{11}, z_{12}\} \).

Now we shall go into \( J^5_m \). By direct computations as in Lemma 5, we have

**Lemma 8.** If \( n < 10 \) or \( n = 12 \), then \( s_n(\alpha) = 0 \) for any \( \alpha \in J^5_n \).

Moreover, we have

**Lemma 9.** Let \( n = 2^s, s > 4 \). Then \( J^5_n \) contains a class \( \alpha \) such that \( s_{2-1,2-1}(\alpha) \equiv 1 \mod 2 \).

**Proof.** For \( s = 4 \), \( RP(7, 0, \ldots, 0) \) (9 zeroes) is as required. For \( s > 4 \),

\[
RP(2^{s-3} - 2, s^{s-3} - 1, \ldots, 2^{s-3} - 1, 0, 0)
\]
is as required.

Let us observe monomials of the generators for \( \mathcal{M}_* \). First, notice that \( J^3_m \subset J^2_m \)
follows from \( [4] \) and Proposition 1. By their definitions, \( y_5, y_6 \in J^2_m \) and \( z_9, z_{10}, z_{12} \in J^2_m \). Furthermore, Proposition 1 shows \( y_5, y_6, x_2^2 \in J^2_m \) and Lemma 9 shows \( x_2^2 \in J^3_m \) for \( r > 3 \). Clearly, \( y_5^2 \in J^4_m \). Now consider \( y_6^2 \). By the examination of the characteristic numbers, we can see that \( y_6^2 \) is the class \( x_2 y_5^2 + \{RP(3, 2, 2, 0, 0, 0, 0)\} \).

Recall that \( RP(\lambda) = RP(3, 2, 2, 0, 0, 0, 0) \) is the projective space bundle over \( M = RP(3) \times RP(2)^2 \times RP(0)^3 \) associated to \( \lambda = \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_6 \rightarrow M \). An involution of \( M \), given by \( (a, b_1, b_2, c_1, c_2, c_3) \rightarrow (a, b_2, b_1, c_1, c_2, c_3) \), induces a fiber
preserving involution $T$ of $RP(\lambda)$; i.e., we can define an involution $T$ of $RP(\lambda)$ by

$$T(u_1, u_2, u_3, u_4, u_5, u_6) = (-u_1, -u_3, -u_2, -u_4, u_5, u_6).$$

It is easy to see that all the components of the fixed point set of $T$ are of codimension 5. Therefore $y_6^2 \in J_4^2$. Referring to the results of [1], we can show that $J_4^2$ contains all monomials of generators for $J_4^2$ except those of the form

$$y_5y_6x(m), \quad z_9x(m), \quad x_4^2x(m), \quad z_{10}x(m), \quad z_{12}x(m).$$

Here, $x(m)$ is the class of $RP(2^r) \times RP(2^{r_2}) \times \cdots \times RP(2^{r_t})$ for $m = 2^{r_1} + 2^{r_2} + \cdots + 2^t, r_1 > r_2 > \cdots > r_t > 0$. By straightforward calculation, we have the tables of characteristic numbers.

<table>
<thead>
<tr>
<th>$W_1^{n-9}W_2^3W_3^3$</th>
<th>$y_5y_6x(n - 11)$</th>
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</thead>
<tbody>
<tr>
<td>$W_1^{n-11}W_2^3W_3^3$</td>
<td>1</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>$W_1^{n-8}W_2^4$</th>
<th>$x_4^2x(n - 8)$</th>
<th>$z_{10}x(n - 10)$</th>
<th>$z_{12}x(n - 12)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 $n \equiv 0, 4$ (8)</td>
<td>0 $n \equiv 0, 2$ (8)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 $n \equiv 2, 6$ (8)</td>
<td>1 $n \equiv 4, 6$ (8)</td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$W_1^{n-10}W_2^2W_3^2$</th>
<th>1</th>
<th>1 $n \equiv 0, 4$ (8)</th>
<th>1 $n \equiv 0, 6$ (8)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0 $n \equiv 2, 6$ (8)</td>
<td>0 $n \equiv 2, 4$ (8)</td>
</tr>
</tbody>
</table>

Using these, together with Lemmas 5, 7 and 8, we can attain our theorem immediately.

**Remark.** As a corollary, we can show $J_{k+1}^{2k+1} \subset J_{k+1}^k$ for every integer $k > 3$.

**References**


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