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L_0 IS ω-TRANSITIVE

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ABSTRACT. Let L_0 be the space of measurable functions on the unit interval. Let F and G be two subspaces of L_0, each isomorphic to the space of all sequences. It is proved that there is a linear homeomorphism of L_0 onto itself which takes F onto G. A corollary of this is a lifting theorem for operators into L_0/F, where F is a subspace of L_0 isomorphic to the space of all sequences.

Let L_0 denote the space of all measurable functions on [0, 1] with the topology of convergence in measure. In [1] it was proved that if F and G are finite-dimensional subspaces of L_0 of the same finite dimension, then there is an isomorphism (linear homeomorphism) of L_0 onto itself which takes F onto G. In this note we prove this result when F and G are isomorphic to ω, the space of all sequences.

THEOREM. *Let F and G be subspaces of L_0 which are isomorphic to ω. Then there is an isomorphism of L_0 onto itself taking F onto G.*

We give a corollary and then prove the theorem. Recall that an F-space X has L_0-structure if, for each $\varepsilon > 0$, X can be written as a topological direct sum $X = \bigoplus_{i=1}^{n} X_i$ where each X_i is a subspace of X and the diameter of X_i is less than ε, for each i (see [1]).

COROLLARY. *Let F be a subspace of L_0 which is isomorphic to ω. Let X be an F-space with L_0-structure, and let T be a linear operator from X into L_0/F. Then there is a unique linear operator $\tilde{T}$ from X to L_0 such that $T = \pi\tilde{T}$, where π is the canonical quotient map from L_0 onto L_0/F. ($\tilde{T}$ is said to be a lifting of T.)*

PROOF OF THE COROLLARY FROM THE THEOREM. We first give some notation and describe a special setting of the corollary which will be useful in the proof of the theorem.

If f is in L_0, we denote by $[f]$ the one-dimensional space spanned by f. The support of f will be denoted by $\operatorname{supp} f$. If A is a measurable subset of [0, 1], we denote by $L_0(A)$ the subset of L_0 consisting of all f such that $\operatorname{supp} f \subset A$. As usual, functions equal almost everywhere are identified, and relations between sets are stated modulo sets of measure zero.

Suppose that (f_i) is a sequence of nonzero elements of L_0 such that the sets $S_i = \operatorname{supp} f_i$ are pairwise disjoint. Let $F = \overline{\operatorname{span}(f_i)}$. It is clear that on F, the

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L_0-topology is equivalent to the topology of convergence in measure on each S_i; hence F is isomorphic to ω. We call a copy of ω obtained in this way a *disjoint* copy of ω.

Keeping the previous notation, we suppose that $F = \overline{\operatorname{span}(f_i)}$ is a disjoint copy of ω. Then if $B = \bigcup_i \operatorname{supp} f_i$ and if $C = [0, 1] \sim B$, we have that L_0/F is the topological product $L_0(C) \oplus \prod_{i=1}^{\infty} L_0(\operatorname{supp} f_i)/[f_i]$, canonically. Suppose $T\colon X \to L_0/F$ is a linear operator. Let q_i be the quotient map of L_0/F onto $L_0(\operatorname{supp} f_i)/[f_i]$, for each i. By [**1**, Theorem 3.6], for each i there is a map $\tilde{T}_i\colon X \to L_0(\operatorname{supp} f_i)$ which is a lifting of the map $q_i T$. Then the map $\chi_C T \oplus \prod_i \tilde{T}_i$ is a lifting of T, as required. The uniqueness of $\tilde{T}$ follows immediately: if T_1 is another lifting of T, then $\tilde{T} - T_1$ maps into the locally convex space F, and so must be identically zero.

Now suppose G is any isomorph of ω in L_0. We have shown that the conclusion of the corollary holds for L_0/F, above. By the theorem, there is an isomorphism of L_0 onto itself which takes F onto G; hence the corollary holds in general.

PROOF OF THE THEOREM.

LEMMA 1. *Let $F = \overline{\operatorname{span}(f_i)}$ and $G = \overline{\operatorname{span}(g_i)}$ be two disjoint copies of ω in L_0. Then there is an isomorphism of L_0 onto itself which takes F onto G.*

PROOF. Let $A = [0, 1] \sim \bigcup_{i=1}^{\infty} \operatorname{supp} f_i$ and let $B = [0, 1] \sim \bigcup_{i=1}^{\infty} \operatorname{supp} g_i$. Let $C = A \cup \operatorname{supp} f_1$ and let $D = B \cup \operatorname{supp} g_1$. By [**1**, Proposition 2.2] there is an isomorphism T_1 of $L_0(C)$ onto $L_0(D)$ taking f_1 onto g_1. For the same reason, if $i \geqslant 2$ there is an isomorphism T_i of $L_0(\operatorname{supp} f_i)$ onto $L_0(\operatorname{supp} g_i)$ taking f_i onto g_i. Putting these isomorphisms together in the obvious way yields the result. Q.E.D.

We next consider an intermediate case (the *linearly independent* case):

Let (e_j) be a sequence in L_0 equivalent to the natural basis of ω. In addition, assume that there is a partition (B_i) of $\bigcup_j \operatorname{supp} e_j$ such that for each i,

$$N_i = \{j \in N\colon (\operatorname{supp} e_j) \cap B_i \neq \varnothing\}$$

is a finite set, and the set of restrictions $\{e_j|_{B_i}\colon j \in N_i\}$ is a linearly independent set. By [**1**, Proposition 2.2], for each i, there is an isomorphism T_i of $L_0(B_i)$ onto itself taking the functions $\{e_j|_{B_i}\colon j \in N_i\}$ onto disjointly supported functions.

Then the maps T_i obviously induce an isomorphism of $L_0(\bigcup_i \operatorname{supp} e_i)$ onto itself taking the e_i's onto disjointly supported functions. This completes the proof in this case, since it has been reduced to the disjoint case.

Turning now to the general case, let (e_i) be a sequence equivalent to the natural basis of ω, with no additional assumptions. We will prove: there is a basis $(\tilde{e}_i)$ of $\overline{\operatorname{span}(e_i)}$ and a sequence (B_i) of pairwise disjoint measurable sets satisfying

(i) each $\tilde{e}_i$ is a finite linear combination of the e_j's;

(ii) for each i, the set $N_i = \{j \in N\colon \operatorname{supp} \tilde{e}_j \cap B_i \neq \varnothing\}$ is a finite set;

(iii) for each i, the set of restrictions $\{\tilde{e}_j|_{B_i}\colon j \in N_i\}$ is linearly independent;

(iv) $\bigcup_i B_i = \bigcup_j \operatorname{supp} \tilde{e}_j$.

Once this has been proved, the argument for the linearly independent case can be applied to the sequences $(\tilde{e}_i)$ and (B_i) and the proof of the theorem will be complete.

We next single out an important property of arbitrary isomorphs of ω in L_0.

LEMMA 2. *Let E be a subspace of L_0 isomorphic to ω, and let (e_i) be a sequence in E corresponding to the usual basis of ω. Then*

$$\lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} \operatorname{supp} e_k\right) = 0,$$

where μ is Lebesgue measure.

PROOF. If the statement is false, there are $\varepsilon > 0$ and a subsequence (n_k) of the positive integers such that

$$\mu\left[\bigcup_{i=n_k}^{n_{k+1}-1} \operatorname{supp} e_i\right] > \varepsilon,$$

for each k. By [2, Lemma 1] for each k we can find scalars $a_{n_k}, a_{n_k+1}, \ldots, a_{n_{k+1}-1}$ such that

$$\operatorname{supp}\left(\sum_{i=n_k}^{n_{k+1}-1} a_i e_i\right) = \bigcup_{i=n_k}^{n_{k+1}-1} (\operatorname{supp} e_i).$$

Set $g_k = \sum_{i=n_k}^{n_{k+1}-1} a_i e_i$. Then $\mu(\operatorname{supp} g_k) > \varepsilon$, so we can choose scalars r_k so large that $\int(|r_k g_k|/(1 + |r_k g_k|))\, d\mu > \varepsilon$. But then we have a contradiction, since $r_k g_k \to 0$. Q.E.D.

Now let the e_i's be as in Lemma 2, and let $\mathcal{F}$ be the family of all finite subsets of the positive integers. For each $F \in \mathcal{F}$, let

$$C_F = \bigcap_{i\in F} \operatorname{supp} e_i \sim \left(\bigcup_{i\notin F} \operatorname{supp} e_i\right).$$

The sets $\{C_F\colon F \in \mathcal{F}\}$ are pairwise disjoint and (the important point), by Lemma 2, $\mu(\bigcup_{i=1}^{\infty} \operatorname{supp} e_i \sim \bigcup_{F\in\mathcal{F}} C_F) = 0$.

Let (C_i) be an enumeration of the nonempty sets among the sets C_F, $F \in \mathcal{F}$.

We shall describe a repetitive procedure for generating the sequence $\tilde{e}_i$. The procedure alternates between two similar steps in such a way that we are sure to consider every e_i and every C_i. During each step, we delete elements from the sequence (e_j) and the enumeration (C_i).

Step $2s - 1$ ($s = 1, 2, \ldots$). Find the smallest subscript m such that e_m has not been deleted from the sequence (e_i). (In the first application of this step, $m = 1$.) Choose n such that $C_n \cap \operatorname{supp} e_m \neq \varnothing$.

Consider the (finite) set

$$P_{2s-1} = \{e_i|_{C_n}\colon i \in F_{2s-1}\}$$

consisting of all restrictions $e_i|_{C_n}$ for which $C_n \cap \operatorname{supp} e_i \neq \varnothing$. (Only e_i's which have not been deleted on a previous step are to be included.) From this set extract a subset $\{e_i|_{C_n}\colon i \in K_{2s-1}\}$ which is a basis for the span of P_{2s-1}.

We require that $e_m|_{C_n}$ be one of the elements of this basis.

Let $\tilde{e}_k = e_k$ for k in K_{2s-1}, and delete those e_k's from the original sequence (e_i). For each j in $F_{2s-1} \sim K_{2s-1}$, let g_j be the linear combination of the functions $\{\tilde{e}_k\colon k \in K_{2s-1}\}$ such that $e_j + g_j \equiv 0$ on C_n. For j in $F_{2s-1} \sim K_{2s-1}$, replace each e_j in the original sequence by $e_j + g_j$ (and relabel it e_j).

Delete C_n from the original enumeration (C_i), and define $A_{2s-1} = C_n$.

Step $2s$ ($s = 1, 2, \ldots$). Find the smallest subscript n such that C_n has not been deleted from the enumeration (C_i). Delete C_n from the enumeration, and define $A_{2s} = C_n$. Choose an m with $C_n \cap \operatorname{supp} e_m \neq \varnothing$. (If there is no such m, set $F_{2s} = K_{2s} = \varnothing$ and terminate this step.) Just as in the odd-numbered step above, consider the set P_{2s}, extract a basis, define the corresponding $\tilde{e}_i$'s, delete those elements from (e_i), and replace other elements in (e_i). This ends step $2s$.

To generate the complete sequence $\tilde{e}_i$, we do step 1, step 2, step 3, ... Notice that after step $2s$,

$$\operatorname{span}\{\tilde{e}_i : \tilde{e}_i \text{ has been defined}\} \supset \operatorname{span}\{e_1, \ldots, e_s\}.$$

Thus

(a) $\operatorname{span}(\tilde{e}_i) = \operatorname{span}(e_i)$.

Also,

(b) $\bigcup_{s=1}^{\infty} A_s = \bigcup_i C_i = \bigcup_i \operatorname{supp} e_i = \bigcup_i \operatorname{supp} \tilde{e}_i$. It is easy to see that

(c) for each s, $\{\tilde{e}_i|_{A_s} : i \in K_s\}$ is a linearly independent set of functions; and

(d) $\tilde{e}_i|_{A_l} \equiv 0$ for $l < s$, $i \in K_s$.

From (c) it follows that for each s there is a sequence $(A_m^s)_{m=1}^{\infty}$ of pairwise disjoint measurable sets which partition A_s and have the property that, for each m, the set of restrictions $\{\tilde{e}_j|_{A_m^s} : j \in K_s\}$ is linearly independent. (This is proved in Lemma 3.) Now define

$$B_i = \bigcup_{j=1}^{i} A_{i-j+1}^j,$$

for each i. Note that $\{B_i\}$ is a partition of $\bigcup_i \operatorname{supp} e_i$. Also note that, for each i, only finitely many of the functions $\tilde{e}_j$ are not identically zero on B_i–namely, those defined in the ith step and possibly some of those defined in earlier steps. Thus conditions (i), (ii), and (iv) are satisfied for the sequences $(\tilde{e}_i)$, (B_i). It remains to check condition (iii).

Let k be an integer and suppose that

$$\sum_{i \in N_k} c_i \tilde{e}_i \equiv 0 \quad \text{on } B_k;$$

we must show that each c_i is zero. Write $N_k = M_1 \cup M_2 \cup \cdots \cup M_k$, where i is in M_j if $\tilde{e}_i$ was defined in step j. Then $\sum_{i \in N_k} c_i \tilde{e}_i|_{A_k^1} \equiv 0$. But on A_k^1 all $\tilde{e}_i$ are zero except those defined in step 1. So

$$\sum_{i \in M_1} c_i \tilde{e}_i|_{A_k^1} \equiv 0;$$

since restrictions to A_k^1 are linearly independent, it follows that $c_i = 0$ if i is in M_1. Next, working on A_{k-1}^2, we obtain that

$$\sum_{i \in M_1 \cup M_2} c_i \tilde{e}_i|_{A_{k-1}^2} \equiv 0,$$

and then $\sum_{i \in M_2} c_i \tilde{e}_i|_{A_{k-1}^2} \equiv 0$, and we conclude similarly that $c_i = 0$ for i in M_2. Proceeding in this way, we obtain that all the c_i's are 0. This shows that the

sequences $(\tilde{e}_i)$, (B_i) have the properties claimed for them and completes the proof of the theorem.

LEMMA 3. *Let A be a measurable set of positive measure in* $[0, 1]$ *and let* $(f_i)_{i=1}^n$ *be linearly independent elements of* $L_0(A)$. *Then there are disjoint measurable subsets* A_1 *and* A_2 *of* A, *each of positive measure, such that* $\{f_i|_{A_1}\}$ *and* $\{f_i|_{A_2}\}$ *are linearly independent sets.*

PROOF. For a measurable set B in A of positive measure, define $\mathfrak{R}_B = \{r = (r_1, r_2, \ldots, r_n): \sum r_i f_i = 0 \text{ a.e. on } B\}$. We first show that given B, there is a measurable $C \subset B$ of positive measure such that if $D \subset C$ and D has positive measure, then $\mathfrak{R}_D = \mathfrak{R}_C$. To see this, given B, choose, if possible, $B_1 \subset B$, B_1 of positive measure, and a vector r^1 in $\mathfrak{R}_{B_1}$, $r^1 \neq 0$. Now choose, if possible, a set $B_2 \subset B_1$, B_2 of positive measure, and a vector r^2 in $\mathfrak{R}_{B_2}$, r^2 independent of r^1. Now choose, if possible, a set $B_3 \subset B_2$, B_3 of positive measure, and r^3 in $\mathfrak{R}_{B_3}$, r^3 independent of r^1 and r^2. Continue. This process must terminate with some B_j, $j \leqslant n$. Thus if $B_j = C$ and $D \subset C$, D of positive measure, then $\mathfrak{R}_D = \mathfrak{R}_C$.

Continuing with the proof of the lemma, we let $\mathcal{Q}$ be the family of all measurable subsets C of A with positive measure and having the property that if D is a measurable subset of C of positive measure, then $R_D = R_C$. The above construction shows that every measurable set in A of positive measure contains a set in $\mathcal{Q}$. Now let $\mathcal{C}$ be a maximal family of pairwise disjoint elements of $\mathcal{Q}$. Then $\mathcal{C}$ is countable. Let (C_i) be an indexing of the elements of $\mathcal{C}$; by maximality, $A \sim \bigcup_i C_i$ has measure zero.

For each i let E_i be any measurable subset of C_i with $0 < \mu(E_i) < \mu(C_i)$ and let $F_i = C_i \sim E_i$. Let $A_1 = \bigcup_i E_i$ and let $A_2 = \bigcup_i F_i$. Suppose r is a nonzero vector in $\mathfrak{R}_{A_1}$. Then $r \in \mathfrak{R}_{E_i}$ for each i, so $r \in R_{C_i}$ and then $r \in \mathfrak{R}_A$, contradicting the linear independence of (f_i) on A. Similarly, $\mathfrak{R}_{A_2}$ contains the 0 vector alone. The proof of the lemma is complete.

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