SPECTRAL INCLUSION FOR DOUBLY COMMUTING SUBNORMAL n-TUPLES

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Abstract. Let \( S = (S_1, \ldots, S_n) \) be a doubly commuting \( n \)-tuple of subnormal operators on a Hilbert space \( \mathcal{H} \) and \( N = (N_1, \ldots, N_n) \) be its minimal normal extension acting on a Hilbert space \( \mathcal{K} \supseteq \mathcal{H} \). We show that \( \text{Sp}(S, \mathcal{H}) \supseteq \text{Sp}(N, \mathcal{K}) \) and \( \text{Sp}(S, \mathcal{H}) \subseteq \text{p.c.h.}(\text{Sp}(N, \mathcal{K})) \), where \( \text{Sp} \) denotes Taylor spectrum and p.c.h. polynomially convex hull.

1. Introduction. Let \( S \) be a subnormal operator on a Hilbert space \( \mathcal{H} \) and \( N \) be its minimal normal extension to a Hilbert space \( \mathcal{K} \supseteq \mathcal{H} \). A well-known result of Halmos [5] asserts that \( \sigma(S) \supseteq \sigma(N) \), where \( \sigma \) denotes spectrum. Bram then proved in [2] that \( \sigma(S) \subseteq \text{p.c.h.}(\sigma(N)) \), the polynomially convex hull of \( \sigma(N) \). (He actually proved more: if \( U \) is any bounded component of \( \mathbb{C} \setminus \sigma(N) \), then \( U \cap \sigma(S) = \emptyset \) or \( U \subseteq \sigma(S) \).

The question arises as to whether the spectral inclusion holds for commuting \( n \)-tuples \( S = (S_1, \ldots, S_n) \) of subnormal operators on \( \mathcal{H} \). A first comment is in order: not every such \( n \)-tuple has a commuting normal extension, i.e., it is not always possible to find a commuting \( n \)-tuple \( N = (N_1, \ldots, N_n) \) of normal operators on a Hilbert space \( \mathcal{K} \supseteq \mathcal{H} \) such that \( N_i \mathcal{K} \subset \mathcal{K} \) and \( N_i|_{\mathcal{H}} = S_i \) for all \( i = 1, \ldots, n \) (see [8] for an example). There are a number of conditions that guarantee the existence of such an extension (see for instance [1, 6, 7 and 9]). We shall call the \( n \)-tuple \( S \) subnormal in case it admits a commuting normal extension. It follows from Bram's paper [2] (combining the corollary on p. 88 with Theorem 8) that any doubly commuting \( n \)-tuple \( S = (S_1, \ldots, S_n) \) (i.e., \( S_i S_j = S_j S_i \) for all \( i, j \) and \( S_i S_j^* = S_j^* S_i \) for \( i \neq j \)) of subnormal operators is subnormal. (Ito [6] has extended this further.) Also, it is clear that a subnormal \( n \)-tuple has a unique, up to isometric isomorphism, minimal normal extension. The \( n \)-tuples to be considered are, therefore, the subnormal ones. We must now agree on the right notion of joint spectrum. First, we need some notation. For an \( n \)-tuple \( T = (T_1, \ldots, T_n) \) of operators on \( \mathcal{K} \), let \( \sigma_d(T) \) denote the right spectrum of \( T \), that is, \( \sigma_d(T) = \{ \lambda \in \mathbb{C}^n : \Sigma_{i=1}^n (T_i - \lambda_i)^*(T_i - \lambda_i)^* \text{ is not invertible} \} \). If \( T \) is commuting and \( \mathcal{A} \) is a Banach algebra containing the \( T_i \)'s in its center, let \( \sigma_d(T) \) and \( \text{Sp}(T, \mathcal{K}) \) denote the spectra of \( T \) with respect to \( \mathcal{A} \) and \( \mathcal{K} \), respectively, i.e., \( \sigma_d(T) = \{ \lambda \in \mathbb{C}^n : \Sigma_{i=1}^n (T_i - \lambda_i)A_i = I \text{ cannot be solved for } A_i \in \mathcal{A} \} \). (For a definition of \( \text{Sp} \) see [4 or 12].)
Janas has shown in [7] that if $S$ is subnormal with minimal normal extension $N$ and $\mathcal{A}$ is a maximal abelian Banach algebra containing the $S_i$'s, then $\sigma_S(N) \supset \sigma(N)$. (As it turns out, there is universal agreement on the right notion of spectrum for a normal $n$-tuple, since $\sigma(N) = \sigma^{(N)}(N) = \sigma^B(N)$ for any abelian C*-algebra $B$ containing the $N_i$'s.) It is a result of Taylor [12, Lemma 2.1] that $\sigma_N(T, \mathcal{C}) \subset \sigma_S(T)$ for any Banach algebra $\mathcal{A}$ whose center contains the $T_i$'s, so that $\sigma_N(S, \mathcal{C}) \supset \sigma(N, \mathcal{C})$ is perhaps the appropriate inclusion to study. One could look for joint spectra smaller than $\sigma$, like those considered by Słodkowski [11]. There are easy examples that show that $\sigma_{n,k}$ ($k < n$) will not do; on the other hand, $\sigma_{n,0} = \sigma$ for a doubly commuting subnormal $n$-tuple [4, Corollary 3.8]. (The notation in the last sentence is from [11].) We have posed in [4] the following question: Does $\sigma_N(S, \mathcal{C}) \supset \sigma(N, \mathcal{C})$? In this paper we give an affirmative answer where $S$ is doubly commuting. Using a result of Janas we also prove that $\sigma_N(S, \mathcal{C}) \subset \text{p.c.h.}(\sigma_N(N, \mathcal{C}))$. Our proof is based on a theorem of Bram's on the commutant of the C*-algebra generated by a subnormal operator, a basic estimate for the left spectrum of an $n$-tuple, the functional calculus for normal $n$-tuples and our characterization of $\sigma$ for doubly commuting $n$-tuples of hyponormal operators (as $\sigma$).

2. A basic fact about the left spectrum. Let $T = (T_1, \ldots, T_n)$ be an $n$-tuple (not necessarily commuting) of operators $\mathcal{C}$ and $\sigma_l(T)$ be the left spectrum of $T$, that is,

$$\sigma_l(T) = \left\{ \lambda \in \mathbb{C}^n : \sum_{i=1}^n (T_i - \lambda_i)^*(T_i - \lambda_i) \text{ is not invertible} \right\}.$$  

Let

$$\delta(T) = \inf \left\{ \|Tx\| = \left( \sum_{i=1}^n \|T_ix\|^2 \right)^{1/2} : \|x\| = 1 \right\}$$

and

$$m_l(T) = \inf \left\{ |\lambda| = \left( \sum_{i=1}^n |\lambda_i|^2 \right)^{1/2} : \lambda \in \sigma_l(T) \right\}.$$ 

The following lemma is probably well known among the specialists. We include a proof for the sake of completeness (see [10] for a different proof).

**Lemma 1.** For an arbitrary $n$-tuple $T$, $m_l(T) > \delta(T)$.

**Proof.** Let $\lambda \in \mathbb{C}^n$ and $x \in \mathcal{C}$, $\|x\| = 1$. Then

$$\sum_{i=1}^n \|(T_i - \lambda_i)x\|^2 = \sum_{i=1}^n \|T_ix\|^2 + \sum_{i=1}^n |\lambda_i|^2 - 2 \sum_{i=1}^n \text{Re}(T_ix, \lambda_ix)$$

and

$$\left| \sum_{i=1}^n \text{Re}(T_ix, \lambda_ix) \right| \leq \sum_{i=1}^n |\lambda_i| \|T_ix\| \leq \left( \sum_{i=1}^n |\lambda_i|^2 \right)^{1/2} \left( \sum_{i=1}^n \|T_ix\|^2 \right)^{1/2}.$$
Thus,

$$\sum_{i=1}^{n} \| (T_i - \lambda_i)x \|^2 > \left[ \left( \sum_{i=1}^{n} \| T_i x \|^2 \right)^{1/2} - \left( \sum_{i=1}^{n} |\lambda_i|^2 \right)^{1/2} \right]^2.$$ 

Therefore, if $|\lambda| < \delta(T)$ then $\sum_{i=1}^{n} \| (T_i - \lambda_i)x \|^2 > (\delta(T) - |\lambda|)^2$, so that $\lambda \notin \sigma_l(T)$, from which the result follows.

**Lemma 2.** Let $N$ be a commuting $n$-tuple of normal operators. Then $m_l(N) = \delta(N)$.

**Proof.** $C^*(N_1, \ldots, N_n) = C(\sigma_l(N))$.

3. *Brann's commutant theorem.*

**Lemma 2 (Theorem 8 in [2]; see also [3, Chapter IV]).** Let $S$ be a subnormal operator on $\mathcal{K}$ with minimal normal extension $N$ on $\mathcal{K} \supset \mathcal{K}$. Let $C^*(N)^\prime$, $C^*(S)^\prime$ and $C^*(P)^\prime$ denote the commutants of the $C^*$-algebras generated by $N$, $S$ and the projection $P$ of $\mathcal{K}$ onto $\mathcal{K}$ ($P = P_\mathcal{K}$). The map

$$C^*(N)^\prime \cap C^*(P)^\prime \rightarrow C^*(S)^\prime,$$

is an isometric $\ast$-isomorphism. Moreover, if $Q \in C^*(S)^\prime$ is a projection, then $\Phi^{-1}(Q)$ is the projection on $\mathcal{K}$ whose range is the closed linear span of the family $\{N^*x : x \in Q\mathcal{K}, \ n > 0\}$, so that $N|_{\Phi^{-1}(Q)}$ is the minimal normal extension of $S|_Q$.

**3. The main result.** The following lemma is the keystone for our proof of the spectral inclusion.

**Lemma 4.** Let $S$ be a subnormal operator on $\mathcal{K}$ with minimal normal extension $N$ on $\mathcal{K} \supset \mathcal{K}$. Let $\tau$ be a positive operator in $C^*(S)^\prime$ and $K = \tau^{-1}(\tau)$ the (positive) operator given by Bram's theorem. Assume that $0 \notin \sigma_l(S, \mathcal{K})$. Then $0 \notin \sigma_l(N, \mathcal{K})$.

**Proof.** By definition of $\sigma_l$, we know that $SS^* + \tau^2$ is invertible, say $SS^* + \tau^2 > 3\varepsilon$ for some $\varepsilon > 0$. Let the positive numbers $t_k$ and projections $Q_k \in C^*(S)^\prime$ $(k = 1, \ldots, m)$ be chosen so that

(i) $\sum_{k=1}^{m} Q_k = I$,

(ii) $Q_k Q_l = 0$ if $k \neq l$, and

(iii) $\| H^2 - \sum_{k=1}^{m} t_k^2 Q_k \| < \varepsilon$.

Then

$$SS^* + \sum_{k=1}^{m} t_k^2 Q_k > 2\varepsilon.$$ 

Since the ranges of the $Q_k$'s reduce $S$ (all $k$), are pairwise orthogonal and span $\mathcal{K}$, we can define $S_k = S|_{Q_k\mathcal{K}}$ acting on $Q_k\mathcal{K}$ and write $(\ast)$ as

$$\bigoplus_{k=1}^{m} (S_k S_k^* + t_k^2) > 2\varepsilon.$$ 

Thus, for each $k$, $S_k S_k^* + t_k^2 > 2\varepsilon$, or $\| S_k^* x \|^2 + t_k^2 > 2\varepsilon$, $x \in Q_k\mathcal{K}$, $\| x \| = 1$. In the notation of Lemma 1 this is $\delta(S_k^*, t_k) > \sqrt{2\varepsilon}$, so that $m_l(S_k^*, t_k) > \sqrt{2\varepsilon}$, too.

Now, by the projection property for the left spectrum,

$$\sigma_l(S_k^*, t_k) = \sigma_l(S_k^*) \times \{ t_k \} = \sigma_l(S_k) \times \{ t_k \}$$
(the horizontal bar denoting complex conjugation). Of course, \( \sigma_r(S_k) = \sigma(S_k) \), because \( S_k \) is subnormal. Then

\[
\sigma(S_k^*; t_k) = \sigma(S_k) \times \{ t_k \} \cup \sigma(N_k) \times \{ t_k \},
\]

by the spectral inclusion theorem for subnormal operators and the fact that \( N_k = N|_{\Phi^{-1}(Q_k)\mathbb{K}} \) is the minimal normal extension of \( S_k \). Therefore, \( m_r(N_k, t_k) > m_r(S_k^*, t_k) > \sqrt{2e} \). By Lemma 2, however, \( \delta(N_k, t_k) = m_r(N_k, t_k) \), so that \( \| N_k x \| ^2 + t_k^2 > 2e, x \in \Phi^{-1}(Q_k)\mathbb{K}, \| x \| = 1. \)

Therefore \( \bigoplus_{k=1}^{n} (N_k^* N_k + t_k^2) > 2e \), or

\[
N^* N + \sum_{k=1}^{m} t_k^2 \Phi^{-1}(Q_k) > 2e.
\]

From (iii) above and the fact that \( \Phi \) is an isometry, we get

\[
\left\| K^2 - \sum_{k=1}^{m} t_k^2 \Phi^{-1}(Q_k) \right\| < e.
\]

This last equation combined with (**) gives \( N^* N + K^2 > e \), as desired.

5. The spectral inclusion theorem. We need one more lemma before we can prove our theorem.

**Lemma 5 (Corollary 3.8 in [4]).** Let \( T = (T_1, \ldots, T_n) \) be a doubly commuting n-tuple of hyponormal operators on \( \mathbb{K} \). Then \( \text{Sp}(T, \mathbb{K}) = \sigma_r(T) \).

**Theorem 1 (spectral inclusion).** Let \( S = (S_1, \ldots, S_n) \) be a doubly commuting subnormal n-tuple on \( \mathbb{K} \) with minimal normal extension \( N = (N_1, \ldots, N_n) \) on \( \mathbb{K} \supset \mathbb{K} \). Then \( \text{Sp}(S, \mathbb{K}) \supset \text{Sp}(N, \mathbb{K}) \).

**Proof.** Assume \( n > 2 \). As in the one-variable case, it is enough to show that \( 0 \notin \text{Sp}(S, \mathbb{K}) \) implies \( 0 \notin \text{Sp}(N, \mathbb{K}) \). Now, if \( 0 \notin \text{Sp}(S, \mathbb{K}) \) and \( H = (\Sigma_{i=2}^{n} S_i S_i^*)^{1/2} \), then \( (S_1, H) \) is right invertible. Let \( T_1^{(1)} = \text{m.e.e.}(S_1) \) acting on \( \mathbb{K}^{(1)} \subset \mathbb{K} \) and \( \Phi_1: C^*(T_1^{(1)})' \cap C^*(P_2)' \to C^*(S_1)' \) be Bram's isomorphism. Let \( T_i^{(1)} = \Phi_1^{-1}(S_i), i = 2, \ldots, n. \) Notice that \( \Phi_1^{-1}(H) = (\Sigma_{i=2}^{n} T_i^{(1)} T_i^{(1)*})^{1/2} \) and that each \( T_i^{(1)} \) is subnormal; actually, \( T_i^{(1)} = N_i |_{\mathbb{K}^{(1)}} \). By Lemma 4, \( (T_1^{(1)}, \Phi_1^{-1}(H)) \) is right invertible, so that \( T_1^{(1)} = (T_1^{(1)}, \ldots, T_n^{(1)}) \) is right invertible, or \( 0 \notin \text{Sp}(T^{(1)}, \mathbb{K}^{(1)}) \), by Lemma 5.

We can now extend \( T_2^{(1)} \) to its minimal norm extension \( T_2^{(2)} \) on \( \mathbb{K}^{(2)} \subset \mathbb{K} \) and repeat the argument so that \( 0 \notin \text{Sp}(T^{(2)}, \mathbb{K}^{(2)}) \). We can continue this process until \( T_n^{(n-1)} \) has been extended. Finally, it is clear that \( \mathbb{K}^{(n)} = \mathbb{K} \) and \( T^{(n)} = N, \) so that \( 0 \notin \text{Sp}(N, \mathbb{K}) \), as desired.

**Remark.** With the notation as in the preceding proof, notice that we actually proved that

\[ \text{Sp}(N, \mathbb{K}) \subset \text{Sp}(T^{(n-1)}, \mathbb{K}^{(n-1)}) \subset \cdots \subset \text{Sp}(T^{(1)}, \mathbb{K}^{(1)}) \subset \text{Sp}(S, \mathbb{K}). \]

**Theorem 2.** Let \( S = (S_1, \ldots, S_n) \) be a subnormal n-tuple on \( \mathbb{K} \) (not necessarily doubly commuting) and \( N = (N_1, \ldots, N_n) \) be its minimal normal extension acting on \( \mathbb{K} \supset \mathbb{K} \). Then \( \text{Sp}(S, \mathbb{K}) \subset \text{p.c.h.}(\text{Sp}(N, \mathbb{K})). \)
Proof (see Janas [7, Corollary to Theorem 5]). Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \text{Sp}(S, \mathcal{H}) \) and \( P(z_1, \ldots, z_n) \) be a polynomial. Then \( P(\lambda) \in P(\text{Sp}(S, \mathcal{H})) = \text{Sp}(P(S), \mathcal{H}) \), by the Spectral Mapping Theorem for Taylor spectrum [13, Theorem 4.8], so that

\[
|P(\lambda)| \leq \sup\{|z|: z \in \sigma(P(S))\} = ||P(S)|| \leq ||P(N)||
\]

\[
= \sup\{|z|: z \in \sigma(P(N))\} = \sup\{|P(z)|: z \in \text{Sp}(N, \mathcal{H})\}.
\]

Thus \( \lambda \in \text{p.c.h.}(\text{Sp}(N, \mathcal{H})) \).

Corollary. Let \( S \) be a doubly commuting subnormal \( n \)-tuple on \( \mathcal{H} \) with minimal normal extension \( N \) on \( \mathcal{H} \supseteq \mathcal{K} \). Assume that \( \text{Sp}(S, \mathcal{K}) \) is polynomially convex. Then \( \text{Sp}(S, \mathcal{H}) = \text{p.c.h.}(\text{Sp}(N, \mathcal{K})) \).

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References

10. N. Salinas, Quasitriangular extensions and problems on joint quasitriangularity (preprint).