A NOTE ON \( L(p, q) \) SPACES AND ORLICZ SPACES WITH MIXED NORMS

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Abstract. Necessary and sufficient conditions are given for the embedding of \( L(p, q) \) spaces and Orlicz spaces with mixed norms.

1. Introduction. Let \( X(0, \infty), Y(0, \infty), Z((0, \infty) \times (0, \infty)) \) be Banach function spaces (cf. [6]) of Lebesgue measurable functions on \((0, \infty)\) (respectively \((0, \infty) \times (0, \infty)\)). Let \( M \) denote the class of real valued Lebesgue measurable functions on \((0, \infty) \times (0, \infty)\) and let \([X, Y] = \{k \in M: \|k\|_{[X,Y]} < \infty\}\), where

\[
\|k\|_{[X,Y]} = \sup \left\{ \int \int |k(x,y)f(y)g(x)| \, dx \, dy : \|f\|_X < 1, \|g\|_Y < 1 \right\}.
\]

Therefore, if \( k \in [X, Y] \) then \( z_k(f) = \int k(x,y)f(y) \, dy \) defines a bounded linear operator \( z_k: X \to Y \).

In the qualitative theory of integral equations (cf. [2]) it is important to determine necessary and sufficient conditions for a continuous embedding \( Z \subseteq [X, Y] \) to hold. It is well known, and easy to see, that \( Z \) is continuously embedded in \([X, Y]\) if and only if \( X \otimes_Y Y' \) is continuously embedded in \( Z' \) (cf. [8]).

In [10] the following results are obtained for tensor products of \( L(p, q) \) spaces and Orlicz spaces (the symbol \( \subseteq \) will be used to denote a continuous embedding).

Theorem A. Let \( A, B \) and \( C \) be Young's functions, then the following statements are equivalent.

(i) \( \exists \theta > 0 \exists A^{-1}(t)B^{-1}(s) < \theta C^{-1}(ts), \forall t, s > 0 \),
(ii) \( L_A \otimes_M L_B \subseteq L_C \),
(iii) \( L_A \otimes_M M(L_B) \subseteq M(L_C) \).

Theorem B. Let \( 1 < p < \infty, 1 < q_i < \infty, i = 1, 2, 3 \). Then,

\[
L(p, q_1) \otimes_M L(p, q_2) \subseteq L(p, q_3)
\]

if and only if the following conditions are satisfied.

(i) \( \max\{q_1, q_2\} < q_3 \),
(ii) \( 1/p + 1/q_3 < 1/q_1 + 1/q_2 \).

In this paper we consider \( L(p, q) \) spaces and Orlicz spaces with mixed norms. Let \( X(Y) \) denote the space defined as follows.

\[
X(Y) = \{ f \in M : \|f\|_{X(Y)} = \|f(x, \cdot)\|_Y < \infty \}.
\]
It is clear that \( X \otimes_{\pi} Y \subseteq X(Y) \) and therefore we have
\[
(1.1) \quad X(Y) \subseteq Z \Rightarrow X \otimes_{\pi} Y \subseteq Z.
\]

We prove the following

**Theorem 1.** Let \( A, B \) and \( C \) be Young’s functions, then the following statements are equivalent.

(i) \( \exists \theta > 0 \exists A^{-1}(t)B^{-1}(s) < \theta C^{-1}(ts), \forall t, s > 0, \)

(ii) \( L_A(L_B) \subseteq L_C, \)

(iii) \( L_A(M(L_B)) \subseteq M(L_C). \)

**Theorem 2.** Let \( 1 < p < \infty, 1 < q_i < \infty, i = 1, 2, 3. \) Then, \( L(p, q_1)(L(p, q_2)) \subseteq L(p, q_3) \) if and only if the following conditions are satisfied.

(i) \( \max\{q_1, q_2\} < q_3, \)

(ii) \( q_1 < p < q_3. \)

The sufficiency of the conditions of Theorem 2 was obtained by Walsh [11].

The reader will be assumed familiar with the theory of \( L(p, q) \) spaces and Orlicz spaces (cf. [10], [5]). We shall follow the notation of [10]. Note, however, that the Marcinkiewicz \( M_A \) spaces of [10] are denoted by \( M(L_A) \) in the present work.

**2. Proof of Theorem 1.** We begin by recalling a construction of A. P. Calderón [1] which plays an important role in the theory of tensor products of function spaces (cf. [8]).

Let \( (\Omega, \mu) \) be a measure space, \( X(\Omega) \) a Banach function space, \( \Sigma(X) \) its unit ball, and \( A \) a Young’s function. For each pair \( (X(\Omega), A) \) define
\[
A^{-1}(X)(\Omega) = \{ f \in M(\Omega) : \exists \lambda > 0 \text{ and } g \in \Sigma(X) \exists |f(x)| < \lambda A^{-1}(|g(x)|) \}
\]
equipped with its natural norm \( A^{-1}(X) \) becomes a Banach function space. Observe that \( A^{-1}(L^1) = L_A \) and \( A^{-1}(L(1, \infty)) \approx M(L_A) \) if \( A \) satisfies the \( \nabla_2 \) condition.

The following result was obtained in [8].

(2.1) **Lemma.** Let \( X, Y \) and \( Z \) be Banach function spaces (as in §1), and \( A, B \) and \( C \) Young’s functions. Moreover, assume that condition (i) of Theorem A is satisfied, then,

(i) \( X \otimes_{\pi} Y \subseteq Z \Rightarrow A^{-1}(X) \otimes_{\pi} B^{-1}(Y) \subseteq C^{-1}(Z), \)

(ii) \( X(Y) \subseteq Z \Rightarrow A^{-1}(X)(B^{-1}(Y)) \subseteq C^{-1}(Z). \)

Now observe that \( L^1(L^1) = L^1 \) and use (2.1) to obtain the equivalence of (i) and (ii). Moreover, since \( L^1(L(1, \infty)) \subseteq L(1, \infty) \), (2.1) implies the equivalence of (i) and (iii), if we put the additional condition that \( B \) and \( C \) satisfy the \( \nabla_2 \) condition. To remove this condition we use the following.

(2.2) **Lemma (cf. [9]).** Let \( X, Y \) and \( Z \) be rearrangement invariant spaces (as in §1). Suppose that there exists a constant \( M > 0 \) such that \( \forall u \in L^1(0, \infty) \) we have
\[
\|\phi_{Y'}(|u|)\|_{X'} \leq M\phi_{Z'}(\|u\|_1),
\]
where \( \phi_{Y'} \) (respectively \( \phi_{Z'} \)) denotes the fundamental function of \( Y' \) (respectively \( Z' \)) (cf. [12]). Then,
\[
X(M(Y)) \subseteq M(Z).
\]
In order to prove the implication (i) \(\Rightarrow\) (iii) we apply (2.2). In fact a simple computation shows that (i) implies \(\phi_{q_2}(s)/\phi_{q_2}(t) < \theta A^{-1}(s/t), \forall t, s > 0.\) Therefore,

\[
\bar{A}\left(\phi_{q_2}(s)/\phi_{q_2}(t)\right) < s/t, \quad \forall t, s > 0.
\]

Let \(u \in L^1(0, \infty),\) then by the above inequality, we have

\[
A\left(\phi_{q_2}(|u(x)|)/\phi_{q_2}(|u|)\right) < |u(x)|/|u| \quad \text{a.e.,}
\]

which readily implies that \(\|\phi_{q_2}(|u|)\|_{L^q} < \theta \phi_{q_2}(|u|)\|_{L^q}.\) Therefore (iii) holds by (2.2).

The reverse implication (iii) \(\Rightarrow\) (i) is trivial and follows from Theorem A and (1.1).

(2.3) Remark. For Orlicz spaces defined on finite measure spaces the equivalence (i) \(\Leftrightarrow\) (ii) was proved in [4]. We point out that similar results hold for Orlicz-Marcinkiewicz spaces defined by generalized Young's functions (cf. [10], [8]).

3. Proof of Theorem 2. For the sake of completeness we prove that the conditions (i) and (ii) are sufficient. Let \(p'\) be defined by \(1/p + 1/p' = 1,\) and let \(u \in L^1(0, \infty),\) then \(\|u\|_p = \|u\|_p'.\) Therefore, by (2.2), \(L^p(L(p, \infty)) \subseteq L(p, \infty)\) (cf. [11]). Suppose now that (i) and (ii) hold, consider two cases: \(q_2 < p\) or \(q_2 > p.\)

In the first case we have

\[
L(p, q_1)(L(p, q_2)) \subseteq L^p \subseteq L(p, q_3).
\]

In the second case we obtain the desired result interpolating (by the complex method) between \(L^p(L(p, \infty)) \subseteq L(p, \infty)\) and \(L^p(L^p) = L^p\) (cf. [5]).

The necessity of (i) follows from (1.1) and Theorem B, while the necessity of (ii) will follow from (3.1) and (3.2) below.

We use some constructions of Cwikel [3].

(3.1) Lemma. \(L(p, q_1)(L(p, q_2)) \subseteq L(p, \infty) \Rightarrow q_1 < p.\)

Proof. Consider two cases: \(q_2 = \infty\) or \(q_2 < \infty.\) In the first case we get \(q_1 < p\) using (1.1) and Theorem B. Suppose now that \(q_2 < \infty\) and \(q_1 > p.\) We shall construct \(f \in L(p, q_1)(L(p, q_2))\) such that \(f \not\in L(p, \infty).\)

Let us choose \(0 < \epsilon < 1\) such that \(p < \epsilon q_1,\) and define (cf. [3])

\[
f(x, y) = \chi_{(0,\epsilon)}(y)F(x), \quad x, y \in (0, \infty),
\]

where \(e(x) = (x + 1)^{-1},\) \(F(x) = \min\{1, [\log(x + 1)]^{-\epsilon/x}\}.\) Then,

\[
\|\|f(x, \cdot)\|_{p, q_1}\|_{p, q_1} \sim \int_{\epsilon - 1}^{\infty} (1 + x)^{-q_1/p}[\log(x + 1)]^{-q_1/p x q_1/p} \frac{dx}{x} < \infty.
\]

On the other hand,

\[
\lambda_f(t) = \{(x, y): 0 < y < \epsilon(x), F(x) > t\}
\]

\[
= \log(\lambda_f(t) + 1).
\]

Therefore, \(f^\ast(t) = F(e^t - 1).\) Thus, for \(t > 1, f^\ast(t) = t^{-\epsilon/p}\) and

\[
\|f\|_{p, \infty} = \sup_{t > 0} \{f^\ast(t) t^{1/p}\} > \sup_{t > 1} \{t^{-\epsilon/p} t^{1/p}\} = \infty.
\]
Lemma 3.2. If \( L(p, q_1)(L(p, q_2)) \subseteq L(p, q_3) \), then \( p < q_3 \).

Proof. Assume without loss that \( q_3 < \infty \). Suppose that \( L(p, q_1)(L(p, q_2)) \subseteq L(p, q_3) \) but to the contrary \( q_3 < p \).

Let us choose \( 0 < \varepsilon < 1 - q_3/p \), and define \( f \) as in (3.1) with
\[
e(x) = e^x, \quad F(x) = e^{-x/p}X_{(0,1)}(x) + e^{-x/p}x^{-1/p-\varepsilon/q}X_{(1,\infty)}(x)
\]
Then, as shown by Cwikel [3], \( \|f\|_{p,q_3} = \infty \). However,
\[
\|f(x, \cdot)\|_{p,q_2} \|_{p,q_1} \sim \int_1^\infty x^{-eq_1/q_3} \frac{dx}{x} < \infty,
\]
a contradiction.

To complete the proof of Theorem 2 proceed as follows. Suppose \( L(p, q_1)(L(p, q_2)) \subseteq L(p, q_3) \), then \( L(p, q_1)(L(p, q_2)) \subseteq L(p, \infty) \). Now apply (3.1), (3.2), (1.1) and Theorem B to obtain the necessity of conditions (i)–(ii).

Remark 3.3. In our announcement [7], condition (ii) of Theorem 2 is incorrectly stated as \( \max(q_1, q_2) < p < q_3 \).

References

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