CONTRACTIVE COMMUTANTS AND INVARIANT SUBSPACES

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Abstract. Let \( T \) be a bounded operator on a Banach space \( \mathcal{X} \) and let \( K \) be a nonzero compact operator. In [1] and [4] it is shown that if \( \lambda \) is a complex number and if \( TK = \lambda KT \), then \( T \) has a hyperinvariant subspace. In [1], S. Brown goes on to show that if \( \mathcal{X} \) is reflexive and if \( TK = \lambda KT \) and \( TB = \mu BT \) for some \( \lambda, \mu \) with \( |\lambda| \neq 1 \) and \( (1 - |\mu|)/(1 - |\lambda|) > 0 \), then \( B \) has an invariant subspace. Below we extend both these results by showing that the entire class of operators satisfying the above conditions on \( B \) has an invariant subspace.

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1. The contractive commutant. Let \( \mathcal{X} \) be an infinite-dimensional Banach space and let \( \mathcal{L} (\mathcal{X}) \) be the algebra of bounded linear operators on \( \mathcal{X} \). The commutant \( \{ T \}' \) of an operator \( T \) is the algebra of operators \( B \) that commute with \( T \). A basic result is the elegant theorem of Lomonosov [5]; the statement below is the distillation by Pearcy and Shields [6].

**Theorem 1.1 (Lomonosov).** If \( \alpha \) is a subalgebra of \( \mathcal{L} (\mathcal{X}) \) with no nontrivial invariant subspaces, and if \( K \) is any nonzero compact operator, then there is an operator \( A \) in \( \alpha \) such that 1 is an eigenvalue of \( AK \).

**Corollary 1.2 (Lomonosov).** If \( \{ T \}' \) contains a nonzero compact operator and \( T \) is not a scalar multiple of the identity, then \( \{ T \}' \) has an invariant subspace.

Let \( \mathcal{C}_c(T) = \{ B \in \mathcal{L}(\mathcal{X}): TB = \lambda BT \text{ for some complex number } \lambda \text{ with } |\lambda| < 1 \} \). Notice that \( \mathcal{C}_c(T) \) is not an algebra, since it fails to be closed under sums. Let \( \{ T \}'_c \) be the (nonclosed) algebra generated by \( \mathcal{C}_c(T) \). We refer to \( \{ T \}'_c \) as the contractive commutant of \( T \). Similarly, let \( \mathcal{C}_\infty(T) = \{ B \in \mathcal{L}(\mathcal{X}): TB = \lambda BT \text{ with } |\lambda| < 1 \} \) and let \( \{ T \}'_\infty \) be the algebra generated by \( \mathcal{C}_\infty(T) \); we call \( \{ T \}'_\infty \) the strictly contractive commutant of \( T \). A number of simple facts are listed below.
Theorem 1.3. (i) If \( A, B \) are in \( \mathcal{C}_c(T) \) (resp. \( \mathcal{C}_{sc}(T) \)), and if \( \mu \in \mathbb{C} \) then \( \mu A \) and \( AB \) are in \( \mathcal{C}_c(T) \) (resp. \( \mathcal{C}_{sc}(T) \)).
(ii) If \( A \in \mathcal{C}_c(T) \) and \( B \in \mathcal{C}_{sc}(T) \) then \( BA \) and \( AB \) lie in \( \mathcal{C}_{sc}(T) \).
(iii) If \( A \in \mathcal{C}_c(T) \) (resp. \( \mathcal{C}_{sc}(T) \)) then \( T^* \in \mathcal{C}_c(A^*) \) (resp. \( \mathcal{C}_{sc}(A^*) \)).
(iv) \( \mathcal{C}_c(T) \) is closed in the weak operator topology.

Proof. (i), (ii), and (iii) are straightforward computations. To prove (iv) we suppose that \( \{B_a\} \) is a net of operators in \( \mathcal{C}_c(T) \) and \( B_a \to B \) weakly. If \( \lambda_a \) is chosen so that \( TB_a = \lambda_a B_a T \), then \( |\lambda_a| < 1 \) for all \( a \) and thus there is a convergent subnet of \( \{\lambda_a\} \); without loss of generality we assume that the entire net \( \{\lambda_a\} \) converges, say to \( \lambda \). Then \( TB_a \) converges weakly to \( TB \), \( \lambda_a B_a T \) converges weakly to \( \lambda BT \), and the result follows.

Lemma 1.4. \( \{T\}_c \) (resp. \( \{T\}_{sc} \)) consists precisely of finite sums, \( \sum_{i=1}^{n} B_i \), where each \( B_i \) lies in \( \mathcal{C}_c(T) \) (resp. \( \mathcal{C}_{sc}(T) \)).

Proof. \( \{T\}_c \) is the algebra generated by \( \mathcal{C}_c(T) \), so clearly every finite sum of operators in \( \mathcal{C}_c(T) \) belongs to \( \{T\}_c \). It is easy to check, using 1.3(i), that the collection of finite sums is an algebra, and thus that it is the same as \( \{T\}_c \). The statement for \( \{T\}_{sc} \) follows similarly.

Corollary 1.5. If \( A \in \{T\}_c \) and \( B \in \{T\}_{sc} \), then \( AB \) and \( BA \) lie in \( \{T\}_{sc} \).

Proof. Use Lemma 1.4 and Theorem 1.3(iii).

The proof of the next result is a slight sharpening of the proof of Theorem 2 of [1].

Theorem 1.6. (i) If \( TB = \lambda BT \) for some complex number \( \lambda \) (not necessarily in the unit disk) then either \( |\lambda| = 1 \) or \( TB \) and \( BT \) are quasinilpotent.
(ii) If \( TK = \lambda KT \) where \( K \) is compact then either \( \lambda \) is a root of unity or \( TK \) and \( KT \) are quasinilpotent.

Proof. (i) It is well known that the nonzero elements of \( \sigma(TB) \) and \( \sigma(BT) \) are the same [3, p. 63]. Thus it follows that \( r(TB) = r(BT) \), where \( r(X) \) denotes the spectral radius of \( X \). Since \( TB = \lambda BT \) we also have \( r(TB) = |\lambda| r(BT) \) and thus \( r(BT) = |\lambda| r(BT) \). Hence either \( |\lambda| = 1 \) or else \( r(BT) \) (and therefore \( r(TB) \)) is 0.
(ii) Suppose \( TK = \lambda KT \) and \( TK \) and \( KT \) are not quasinilpotent. By part (i), \( |\lambda| = 1 \). Let \( 0 \neq z \in \sigma(KT) \). Then \( \lambda z \in \sigma(TK) = \sigma(KT) \). By induction, \( \lambda^n z \in \sigma(KT) \) for all nonnegative integers \( n \). However, \( KT \) is compact and its spectrum cannot contain an infinite set of numbers whose absolute values are bounded away from 0. Thus \( \{\lambda^n z\}_{n=0}^{\infty} \) is a finite set and \( \lambda \) must be a root of unity.

2. Invariant subspaces. The contractive commutant contains the commutant; thus it is less likely that the former should have nontrivial invariant subspaces. The following example shows that \( \{T\}_c \) may indeed be transitive.

Example 2.1. Let \( \mathcal{X} \) be a two-dimensional Hilbert space and let \( T, K_1 \) and \( K_2 \) be defined by

\[
T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad K_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad K_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]
Then $K_1$ and $K_2$ both lie in $C_c(T)$, but it is easy to see that no subspace is invariant for $T, K_1,$ and $K_2$. Notice that in this case, $(T)'_c = \mathcal{L}(\mathcal{H})$.

For an infinite-dimensional example, let $\mathcal{H}$ be any Hilbert space and let $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. Let $T = (0 - I) \in \mathcal{L}(\mathcal{H})$. $C_c(T)$ contains all operators of the form

$$
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix},
\begin{pmatrix}
0 & C \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & D
\end{pmatrix}.
$$

Thus $(T)'_c = \mathcal{L}(\mathcal{H})$ in this case as well. Observe that if $(T)'_c$ were transitive and not dense in $\mathcal{L}(\mathcal{H})$ we would have a solution to the transitive algebra problem.

Our central result shows that under certain conditions $(T)'_c$ and $(T)'_{sc}$ do have invariant subspaces.

**Theorem 2.2.** Let $T$ be a nonzero operator in $\mathcal{L}(\mathcal{H})$. If $(T)'_{sc}$ (resp. $(T)'_c$) contains a nonzero compact operator, then $(T)'_c$ (resp. $(T)'_{sc}$) has a nontrivial invariant subspace.

**Proof.** Let $(T)'_{sc}$ contain a nonzero compact operator $K$. Note that ker $F$ is an invariant subspace for $\mathcal{C}(T)$ and hence for $(T)'_c$; we therefore assume that ker $T = \{0\}$. Suppose that $(T)'_c$ is a transitive algebra. Theorem 1.1 guarantees the existence of an operator $B$ in $(T)'_c$ and a nonzero vector $x$ such that $BKx = x$. By Corollary 1.5, $BK \in (T)'_{sc}$ and thus there exist $B_1, \ldots, B_n \in C_c(T)$ such that $\sum_{i=1}^n B_i = BK$. Let $TB_i = \lambda_i B_i T$, where $|\lambda_i| < 1$ for each $i$. Then $TBK = \sum TB_i = (\sum \lambda_i B_i)T$ and inductively $T^mBK = (\sum_{i=1}^n \lambda_i^m B_i)T^m$ for each positive integer $m$. Hence $T^n x = T^m BK x = (\sum_{i=1}^n \lambda_i^m B_i)T^m x$. We have assumed that $T$ has trivial kernel and thus $T^m x \neq 0$ for every $m$, and it follows that $1$ lies in the point spectrum of $\sum_{i=1}^n \lambda_i^m B_i$ for every $m$. However, this would imply that $1 < \|\sum_{i=1}^n \lambda_i^m B_i\| < \sum_{i=1}^n |\lambda_i|^m \|B_i\|$ for all $m$, which is obviously impossible since $|\lambda_i| < 1$ for all $i$. Hence the assumption that $(T)'_c$ is transitive must be false.

The proof of the other part of the theorem is virtually identical and is omitted.

The corollary is a generalization of Theorem 3 of [1].

**Corollary 2.3.** Suppose that $\mathcal{K}$ is reflexive and that $TK = \lambda KT$ for $K$ a nonzero compact operator, $T$ nonzero, and $|\lambda| \neq 1$. Let $a$ be the algebra generated by all operators $B$ such that $TB = \mu BT$ for some complex number $\mu$ for which $(1 - |\mu|)/(1 - |\lambda|) > 0$. Then $a$ has an invariant subspace.

**Proof.** The theorem covers the case $|\lambda| < 1$. If $|\lambda| > 1$ then $T^* K^* = \lambda^{-1} K^* T^*$ and $K^* \in C_c(T^*)$. The theorem then shows that $(T^*)'_c$ has an invariant subspace. Note that $a = \{B : B^* \in (T^*)'_c\}$. Hence $a^*$ has an invariant subspace, and because of the reflexivity of $\mathcal{K}$ so does $a$.

**Question.** Is it possible to show the existence of an invariant subspace for $(T)'_c$ under the weaker assumption that the closure (in some appropriate topology) of $(T)'_{sc}$ contains a nonzero compact operator? A reasonable first step might be to show that the weaker condition yields a hyperinvariant subspace for $T$.

We remark that C. K. Fong [2] has recently obtained some related results concerning common invariant subspaces of $T$ and $K$, under more general conditions than those discussed here.
REFERENCES


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