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ON A THEOREM FROM "SKEW FIELD CONSTRUCTIONS"

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ABSTRACT. Let F be a skew field and C a central subfield, then the free F-field on X centralizing C is denoted by $F_C \langle\!\langle X \rangle\!\rangle$. The object is to prove the following theorem. Let F be a skew field with a central subfield C, let E be a subfield of F and put $k = E \cap C$; then there is a natural embedding of $E_k \langle\!\langle X \rangle\!\rangle$ in $F_C \langle\!\langle X \rangle\!\rangle$ if and only if E and C are linearly disjoint over k.

This result replaces the erroneous Theorem 6.3.6 on p. 148 of the author's *Skew field constructions*, a counterexample to the latter (due to G. M. Bergman) is also described. The paper also includes an improved form of the specialization lemma (l.c.).

1. In [3] the following result was stated (Theorem 6.3.6, p. 148):

Let F be a (skew) field with centre C, let E be a subfield of F and put $k = E \cap C$, then there is a natural embedding

$$E_k \langle\!\langle X \rangle\!\rangle \to F_C \langle\!\langle X \rangle\!\rangle. \tag{1}$$

We recall that for any set X, $F_C\langle X \rangle$ is the free F-ring on X centralizing C; this is a fir and its universal field of fractions is denoted by $F_C \langle\!\langle X \rangle\!\rangle$.

Unfortunately the above theorem is false as stated; I am indebted to G. M. Bergman for pointing this out to me, as well as supplying me with a counterexample (cf. §5 below). Our object here is to modify the statement so as to obtain a correct result.

We shall use the term 'field' in the sense of 'not necessarily commutative division ring', and sometimes add 'skew' for emphasis. Given subsets X, Y of a field, we say that X *centralizes* Y if $xy = yx$ for all $x \in X, y \in Y$. By a *central* subfield of F we understand a subfield centralizing F. Given a skew field F, let C, D be subfields of F centralizing each other and put $C \cap D = k$, then there is a natural homomorphism

$$C \otimes_k D \to F, \tag{2}$$

obtained by mapping $\sum u_i \otimes v_i$ to $\sum u_i v_i$ ($u_i \in C$, $v_i \in D$). We shall say that C and D are *linearly disjoint* in F over k, if (2) is injective; this reduces to the usual definition when F is commutative.

We shall prove

THEOREM 1. *Let F be a skew field with central subfield C, let E be a subfield of F and put $k = E \cap C$; then there is a natural embedding* (1) *if and only if E and C are linearly disjoint in F over k.*

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The sufficiency will be proved in §4 and the necessity in §5, while §6 lists the modifications needed in the applications made in [3], as well as another form of the specialization lemma which is an easy consequence of Theorem 1. In §2 we briefly recall the concepts and results needed in the proofs and in §3 we give some auxiliary lemmas.

2. We recall that a square matrix A, say $n \times n$, over any ring R is said to be *full* if it cannot be written as $A = PQ$, where P is $n \times r$, Q is $r \times n$ and $r < n$. E.g. any invertible matrix over a field is full, and conversely, a full matrix over a field is a nonzero divisor, hence invertible. Any ring homomorphism clearly maps any nonfull matrix to a matrix which is again nonfull; if moreover, it maps full matrices to full matrices, it is called *honest*. The ring of all $n \times n$ matrices over a ring R is denoted by $\mathfrak{M}_n(R)$.

If K is any ring, then by a *K-ring* we understand a ring R with a homomorphism $K \to R$; when R is a field we speak of a *K-field*. Let R be a *semifir* (i.e. every finitely generated left or right ideal in R is free, of unique rank, cf. [1]); then R has a universal field of fractions U, which is obtained by formally making all the full matrices over R invertible. In particular, if K is a skew field and C a central subfield, then the free K-ring on a set X, $K_C\langle X\rangle$, defined as the K-ring generated by X with defining relations $\alpha x = x\alpha$ for all $x \in X$, $\alpha \in C$, is a semifir. It is even a *fir* (i.e. all left or right ideals are free, of unique rank, cf. [1]), and its universal field of fractions is $K_C \langle\!\langle X \rangle\!\rangle$. Given a homomorphism f between semifirs R, S with universal fields of fractions U, V respectively, we have a diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ U & & V \end{array}$$

and from the construction of U and V it is clear that there is a homomorphism $U \to V$ to complete the diagram to a commutative square if and only if f is honest.

3. The last remark in §2 shows that to prove the sufficiency in Theorem 1 we need only show that under the given condition the natural homomorphism

$$E_k\langle X\rangle \to F_C\langle X\rangle$$

is honest, but this does not seem easy to verify directly. We shall in fact proceed differently, by building up F from $E \otimes_k C$. For this purpose we need several lemmas. Throughout, k is a commutative field.

LEMMA 1. *Let R be a k-algebra and E/k a finite-dimensional field extension. If R is a right Ore domain and $R_E = R \otimes_k E$ is an integral domain, then R_E is again a right Ore domain.*

For a related result see [5, Lemma 1].

PROOF. Let K be the field of fractions of R and consider K_E. This is finite-dimensional over K, because $[K_E : K] = [E : k]$, so if we can prove that it is a domain, it must be a field. Write $R^* = R \setminus \{0\}$, then every element of K_E has the form $p = \sum \lambda_i \otimes a_i b^{-1}$, where $\lambda_i \in E$, $a_i \in R$, $b \in R^*$. Hence $p = ub^{-1}$, where $u = \sum \lambda_i \otimes a_i \in R_E$. It follows that R^* is a right denominator set in R_E (cf. [**1**, Chapter 0] or [**2**, Chapter 12]): Given $u \in R_E$, $b \in R^*$, we have $b^{-1}u = u_1 b_1^{-1}$ for some $u_1 \in R_E$, $b_1 \in R^*$, hence $ub_1 = bu_1$ and clearly $u_1 \neq 0$ if $u \neq 0$. It follows that K_E is a domain: given $ub^{-1}, vc^{-1} \neq 0$, in K_E, we have $b^{-1}v = v_1 b_1^{-1}$ say, hence $ub^{-1} \cdot vc^{-1} = uv_1 b_1^{-1} c^{-1} = uv_1(cb_1)^{-1}$ and this is not 0 because $uv_1 \neq 0$. Thus K_E is a domain, hence a field, and it follows that R_E is a right Ore domain.

LEMMA 2. *Let E be a skew field which is a k-algebra, and let C/k be a commutative field extension. If $E \otimes_k C$ is a domain, then* (i) *$E \otimes_k C$ is an Ore domain and* (ii) *we have*

$$(E \otimes_k C)_C\langle X\rangle \cong E_k\langle X\rangle \otimes_k C. \tag{1}$$

PROOF. To prove (i) we may without loss of generality assume that C is finitely generated over k, say C is a finite extension of $C_0 = k(T)$, where $T = \{t_1, \ldots, t_r\}$ is a finite set of indeterminates. Now $E \otimes_k C_0$ is a localization of $E \otimes k[T] = E[T]$, which is Noetherian (by the Hilbert basis theorem), hence Ore, so $E \otimes C_0$ is an Ore domain. Now $E \otimes_k C = (E \otimes_k C_0) \otimes_{C_0} C$ is a finite-dimensional extension, hence an Ore domain by Lemma 1.

To prove (ii) we remark that we have a k-bilinear map from $E_k\langle X\rangle$, C to $(E \otimes_k C)_C\langle X\rangle$, hence a homomorphism from right to left in (1), and an $(E \otimes_k C)$-ring homomorphism from left to right, by the universal property of the free $(E \otimes_k C)$-ring, and these two maps clearly are mutually inverse.

Next we give a criterion for a skew field to split under a finite field extension. This is a useful result which is probably well known, but no convenient reference seems to be available.

LEMMA 3. *Let E be a skew field which is a k-algebra, and let $C = k(\alpha)$ be a simple algebraic extension of k, generated by an element α with minimal polynomial f over k. Then $E \otimes_k C$ is Artinian and moreover* (i) *it is simple if and only if f is irreducible over the centre of E,* (ii) *it is a field if and only if f is irreducible over E.*

PROOF. It is clear that $E \otimes_k C$ is Artinian, as finite-dimensional E-ring. Now we have $E \otimes_k C \cong E[t]/(f)$, and the 2-sided ideals of $E[t]$ are generated by invariant elements of $E[t]$, which (up to unit factors) are monic polynomials in t with coefficients in the centre of E (cf. [**1**, p. 297]). Thus $E[t]/(f)$ is simple precisely when f is irreducible over the centre, Z say, of E. Suppose now that f is irreducible over Z, thus it is an I-atom in the terminology of [**1**]; then by Proposition 6.5.2, p. 228 of [**1**], we have $E[t]/(f) \cong \mathfrak{M}_n(D)$, where D is a field and n is the number of factors of f in a complete factorization over $E[t]$. Thus $E \otimes_k C$ is a field if and only if $n = 1$, i.e. f is irreducible over E.

Next we show that irreducibility is preserved by free extensions of the ground field.

LEMMA 4. *Let E be a skew field which is a k-algebra. If C/k is a simple algebraic extension whose generator has minimal polynomial f, and f is irreducible over E, then f remains irreducible over $E_k \langle\!\langle X \rangle\!\rangle$, and $R = E_k \langle\!\langle X \rangle\!\rangle \otimes_k C$ is the universal field of fractions of $(E \otimes_k C)_C \langle X \rangle$.*

PROOF. By Lemma 3, $E \otimes_k C$ is a field, and if we can prove the last assertion of the lemma, the rest will follow by another application of Lemma 3.

By Lemma 2 we have the isomorphism (1), and $E \otimes_k C$ is a field, hence $E_k\langle X \rangle \otimes_k C$ is a fir; by inverting certain matrices (viz. all the full matrices over $E_k\langle X \rangle$) we obtain R. But we obtain the universal field of fractions, U say, of $E_k\langle X \rangle \otimes_k C$ by inverting *all* the full matrices; thus U is obtained from R by inverting certain matrices, and these must be full over R since any invertible matrices over U are full. Now $E_k \langle\!\langle X \rangle\!\rangle$ is a field with centre k (cf. [**4**, Theorem 4.3]), and R is obtained from it by tensoring with a finite extension C of k, hence $R = \mathfrak{M}_n(D)$, where D is a field (cf. [**2**, Theorem 2, Corollary of 10.7]). Since $\mathfrak{M}_n(D)$ is simple, the natural homomorphism $\mathfrak{M}_n(D) = R \to U$ is injective, hence $n = 1$, so $R = \mathfrak{M}_n(D) = U$, i.e. R is the universal field of fractions of $(E \otimes_k C)_C\langle X \rangle$, as claimed.

4. We now come to the proof of Theorem 1. We begin by proving a special case:

LEMMA 5. *Let F be a skew field with subfields C, D such that $C \subseteq D \subseteq F$ and C is central in F. Then*

$$F_C \langle\!\langle X \rangle\!\rangle \cong F \overset{\circ}{_D} D_C \langle\!\langle X \rangle\!\rangle, \tag{1}$$

where $^\circ$ *denotes the field coproduct* (*cf.* [**3**, *Chapter* 5]). *Hence we have a natural embedding*

$$D_C \langle\!\langle X \rangle\!\rangle \to F_C \langle\!\langle X \rangle\!\rangle. \tag{2}$$

PROOF. Using * to denote the ring coproduct, we have

$$F \overset{*}{_D} D_C\langle X \rangle = F \overset{*}{_D} D \overset{*}{_C} C\langle X \rangle = F \overset{*}{_C} C\langle X \rangle = F_C\langle X \rangle.$$

By Lemma 5.4.1 of [**3**] it follows that the natural homomorphism $F_C\langle X \rangle \to F \overset{*}{_D} D_C \langle\!\langle X \rangle\!\rangle$ is honest, hence we have an embedding

$$F_C \langle\!\langle X \rangle\!\rangle \to F \overset{\circ}{_D} D_C \langle\!\langle X \rangle\!\rangle,$$

where the field coproduct is by definition the universal field of fractions of the ring coproduct. But the right-hand side is a field generated by F and X, hence it is generated by the image of $F_C \langle\!\langle X \rangle\!\rangle$, i.e. we have the natural isomorphism (1), and (2) is an immediate consequence.

Now let E be a skew field which is a k-algebra and let C/k be a commutative field extension. Assume that $E \otimes_k C$ is an integral domain, then by Lemma 2 it is an Ore domain and so has a field of fractions D. We claim that in this situation we have a homomorphism

$$E_k \langle\!\langle X \rangle\!\rangle \otimes_k C \to D_C \langle\!\langle X \rangle\!\rangle. \tag{3}$$

We remark that such a homomorphism is necessarily an embedding because $E_k \langle\!\langle X \rangle\!\rangle$ has centre k [**4**, l.c.], so the left-hand side of (3) is simple, by Theorem 2, Corollary of 10.7 in [**2**], and the kernel of (3) is therefore zero.

To prove (3) we can proceed stepwise. Thus if (3) holds as stated and also with E, k, C replaced by D, C, C_1 respectively, where $k \subseteq C \subseteq C_1$, then on writing D_1 for the field of fractions of $D \otimes_C C_1$ we have a homomorphism

$$(E_k \langle\!\langle X \rangle\!\rangle \otimes_k C) \otimes_C C_1 \to D_C \langle\!\langle X \rangle\!\rangle \otimes_C C_1 \to D_{1C_1}\langle X \rangle,$$

hence we have a mapping

$$E_k \langle\!\langle X \rangle\!\rangle \otimes_k C_1 \to D_{1C_1} \langle\!\langle X \rangle\!\rangle.$$

Next if C is the union of a directed family of fields C_λ and D_λ is the field of fractions of $E \otimes_k C_\lambda$, and we have maps

$$E_k \langle\!\langle X \rangle\!\rangle \otimes_k C_\lambda \to D_{\lambda C_\lambda} \langle\!\langle X \rangle\!\rangle,$$

then we have honest homomorphisms $E_k\langle X \rangle \to D_{\lambda C_\lambda}\langle X \rangle$, and by passing to the limit, we obtain an honest homomorphism

$$E_k\langle X \rangle \to D_C\langle X \rangle,$$

which gives rise to an embedding of $E_k \langle\!\langle X \rangle\!\rangle$ in $D_C \langle\!\langle X \rangle\!\rangle$, and hence to a homomorphism (3).

Now any commutative field extension of k can be obtained as the direct union of finitely generated extensions, and these can be built up as a succession of simple extensions, either transcendental or algebraic. Taking the transcendental case first, suppose that $C = k(t)$, where t is a central indeterminate. Then (3) reduces to

$$E_k \langle\!\langle X \rangle\!\rangle \otimes_k k(t) \to E(t)_{k(t)} \langle\!\langle X \rangle\!\rangle. \tag{4}$$

By Lemma 6.3.4 of [3], the homomorphism $E_k\langle X \rangle \to E(t)_{k(t)}\langle X \rangle$ is honest, hence we have an embedding of $E_k \langle\!\langle X \rangle\!\rangle$ in $E(t)_{k(t)} \langle\!\langle X \rangle\!\rangle$. Now it follows that we have a bilinear map giving rise to a homomorphism (4). Next, if C is a simple algebraic extension of k, say $C = k(\alpha)$, where α has the minimal polynomial f over k, and $E \otimes_k C$ is assumed to be an integral domain, then f remains irreducible over E (Lemma 3), hence it remains irreducible over $E_k \langle\!\langle X \rangle\!\rangle$ (Lemma 4), therefore $E_k \langle\!\langle X \rangle\!\rangle \otimes_k C$ is a field. Moreover, it is the universal field of fractions of $(E \otimes_k C)_C\langle X \rangle$, by Lemma 4, hence we obtain an isomorphism

$$E_k \langle\!\langle X \rangle\!\rangle \otimes_k C \to (E \otimes_k C)_C \langle\!\langle X \rangle\!\rangle.$$

So (3) is proved in this case also, and it follows that (3) holds generally whenever $E \otimes_k C$ is an integral domain.

To prove the sufficiency in Theorem 1 we take F, C, E, k as in Theorem 1 and assume that E and C are linearly disjoint over k. Then $E \otimes_k C$ is isomorphic to a subring of F, hence an integral domain, which must be Ore by Lemma 2. Let D be the field of fractions, then by (3) we have an embedding

$$E_k \langle\!\langle X \rangle\!\rangle \to D_C \langle\!\langle X \rangle\!\rangle. \tag{5}$$

Since the field of fractions of an Ore domain is unique (up to isomorphism), D may be embedded in F, and by Lemma 5 we have an embedding of $D_C \langle\!\langle X \rangle\!\rangle$ in $F_C \langle\!\langle X \rangle\!\rangle$. Combining this with (5), we obtain the required embedding of $E_k \langle\!\langle X \rangle\!\rangle$ in $F_C \langle\!\langle X \rangle\!\rangle$.

5. To prove the necessity of the conditions, suppose that we have a natural embedding

$$E_k \langle\!\langle X \rangle\!\rangle \to F_C \langle\!\langle X \rangle\!\rangle,$$

then we have a bilinear map of the pair $E_k \langle\!\langle X \rangle\!\rangle$, C into $F_C \langle\!\langle X \rangle\!\rangle$, hence there is a homomorphism

$$(1) \qquad E_k \langle\!\langle X \rangle\!\rangle \otimes_k C \to F_C \langle\!\langle X \rangle\!\rangle.$$

Now $E_k \langle\!\langle X \rangle\!\rangle$ has centre k, so as before we can conclude that (1) is an embedding; the restriction to $E \otimes_k C$ gives a natural embedding of the latter in F, hence E and C are linearly disjoint in F over k.

To give a concrete example (due to G. M. Bergman), let us take for F the real quaternions and put $C = \mathbf{R}$, the centre of F. Now let $\alpha = a + bi$ be a complex number whose real part is not rational, such that $\mathbf{Q}(\alpha) \cap \mathbf{R} = \mathbf{Q}$, e.g., if a, b are independent transcendentals. We put $E = \mathbf{Q}(\alpha)$, so that $k = \mathbf{Q}$.

For any $x \in X$ we can in $F_C \langle\!\langle X \rangle\!\rangle$ write $x = x_0 + x_1$, where $x_0 = \frac{1}{2}(x - ixi)$, $x_1 = \frac{1}{2}(x + ixi)$, then $x_0 i = ix_0$, $x_1 i = -ix_1$. Hence $\alpha x - x\alpha = \alpha x_1 - x_1 \alpha = 2bix_1$; similarly

$$\alpha^2 x - x\alpha^2 = 2abix_1$$

(because $\alpha^2 = a^2 - b^2 + 2abi$), therefore $(\alpha x - x\alpha)^{-1}(\alpha^2 x - x\alpha^2)$ is central in $F_C \langle\!\langle X \rangle\!\rangle$ and lies in the E-field generated by x, but does not lie in k.

6. The applications made of Theorem 6.3.6 in [3] were as follows. On p. 156 the theorem is used twice; the first time we can use Lemma 6.3.4 instead (as was observed l.c.), while the second time we need only the special case covered by Lemma 5 above, so these applications are unaffected. On p. 157, Theorem 6.3.6 is again invoked, but is not strictly applicable because X is changed. In any case there is a direct proof, which in essence goes as follows: We want to obtain an embedding $F_C \langle\!\langle X \rangle\!\rangle \to F_C \langle\!\langle X' \rangle\!\rangle$, where $X \subseteq X'$. This amounts to showing that the homomorphism $F_C\langle X \rangle \to F_C\langle X' \rangle$ is honest, and this follows because $F_C\langle X \rangle$ is a retract, for if a matrix A over $F_C\langle X \rangle$ can be written as $A = PQ$ over $F_C\langle X' \rangle$, where P is $n \times r$, Q is $r \times n$ and $r < n$, then we can put $x = 0$ for $x \in X' \setminus X$ to pull the factorization down to $F_C\langle X \rangle$. On p. 158 the reference to Theorem 6.3.6 should be replaced by a reference to Theorem 6.4.2.

The principal application of Theorem 6.3.6 is in the proof of Theorem 7.2.7, p. 171 of [3], and here the statement has to be modified as follows:

THEOREM 2. *Let E be a skew field with centre C such that* (i) $[E : C] = \infty$ *and* (ii) *C is infinite. If D is a subfield of E, linearly disjoint from C in E over $k = D \cap C$, then every element of $D_k \langle\!\langle X \rangle\!\rangle$ is nondegenerate on E.*

The proof is as for Theorem 7.2.7, using Theorem 1 in place of Theorem 6.3.6. In the proof of Theorem 7.2.6 there is also a reference to Theorem 6.3.6, but this may be replaced by a reference to Lemma 6.3.5 or proved directly, using power series.

Finally we observe the following version of the specialization lemma, which may be proved by Theorem 1.

THEOREM 3. *Let K be a field with centre C such that* (i) $[K : C] = \infty$, (ii) *C is infinite. If E is any subfield of K linearly disjoint from C in K over $k = E \cap C$, then any full matrix over $E_k\langle X\rangle$ is nonsingular for some set of values of X in K.*

The proof follows by observing that the natural map from $E_k\langle X\rangle$ to $K_C\langle X\rangle$ is honest, by Theorem 1.

Theorem 3 is essentially the version of the specialization lemma used in the proof of Theorem 2. The case $E = k$ of Theorem 3 is perhaps worth stating separately:

COROLLARY. *Let K be a field with centre C such that $[K : C] = \infty$ and C is infinite. If E is a subfield of K containing C, then any full matrix over $E_C\langle X\rangle$ is nonsingular for some set of values in K.*

We remark that for any field E which is a k-algebra we can find an extension field K such that $[K : k] = \infty$ and k is the exact centre of K, e.g. $K = E_k \langle\!\langle X \rangle\!\rangle$, as remarked earlier [**4**, Theorem 4.3].

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